

# Computability Theory, Mastermath Fall 2014.

Exam Monday 15.12.2014, 10:00–13:00.

**Problem 1.** Show that there is no universal primitive recursive function. That is, there is no primitive recursive function  $\lambda n, x. F(n, x)$  such that  $f$  is primitive recursive iff there exists  $n$  such that  $f(x) = F(n, x)$  for every  $x$ .

**Problem 2.** (a) Show, using the S-m-n-theorem, that there is a computable function  $f$  such that for all  $x$ ,  $W_{f(x)} = W_x \cup \{x\}$ .

(b) Let  $K$  be the diagonal halting problem. Prove that for every  $x$ ,

$$W_x \subseteq \bar{K} \Rightarrow x \in \bar{K} - W_x.$$

(c) Prove that  $K$  is not simple. [Remember that  $A$  is simple if  $A$  is c.e. and  $\bar{A}$  is infinite but doesn't contain an infinite c.e. set.]

**Problem 3.** Let  $C = \{e \mid \varphi_e \text{ is constant } 0\}$ .

(a) Prove that  $C \in \Pi_2^0$ .

(b) Show that  $C$  is  $\Pi_2^0$ -complete.

**Problem 4.** (a) One of the following  $\lambda_{\rightarrow}^{\text{cu}}$ -terms has a type in  $\lambda_{\rightarrow}^{\text{cu}}$  and the other not. Find a type  $A$  and a proof for one of the following terms that it has that type; argue that the other one doesn't have a type.

$$\lambda xy.(xy)x \text{ and } \lambda xy.x(yx).$$

(b) One of the following simple types has an inhabitant in  $\lambda_{\rightarrow}^{\text{cu}}$  and the other not. Find a  $\lambda$ -term  $M$  and a proof showing that it is of one of these types and argue that the other type has no closed inhabitants.

$$(\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma) \text{ and } (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \alpha).$$

**Problem 5.** Remember that for  $M \in \Lambda$  its code number (for some effective enumeration) is denoted by  $\#M \in \mathbb{N}$ . We write  $\ulcorner M \urcorner \triangleq \mathbf{c}_{\#M}$ , where  $\mathbf{c}_n \triangleq \lambda f x. f^n x$  are the Church's numerals. We may assume that  $\#\mathbf{c}_n > n$  for all  $n \in \mathbb{N}$ .

(a) Prove that there exists an  $M \in \Lambda$  such that

$$M =_{\beta} \ulcorner M \urcorner.$$

[Hint. Apply the second fixed point theorem to 1.]

(b) Show that there exists an  $M$  with

$$M =_{\beta} \ulcorner \ulcorner M \urcorner \urcorner.$$

(c) Show that the following is impossible.

$$\ulcorner M \urcorner =_{\beta} \ulcorner \ulcorner M \urcorner \urcorner$$

[Hint. Show  $\forall n, m \in \mathbb{N}. [\mathbf{c}_n =_{\beta} \mathbf{c}_m \Rightarrow n = m]$  and use the assumption on #.]

**Problem 6.** (a) Argue that for  $M, N \in \Lambda$  the (one step) reduction relation

$$M \rightarrow_{\beta} N$$

after coding is decidable. (An intuitive argument suffices.)

(b) Prove (using (a)) that for  $M, N \in \Lambda$  the (many step) reduction relation

$$M \rightarrow_{\beta} N$$

after coding is c.e.. [Hint.  $M \twoheadrightarrow_{\beta} N$  iff there exists a sequence (with  $n \geq 0$ )

$$M \equiv M_0 \rightarrow_{\beta} M_1 \rightarrow_{\beta} \cdots \rightarrow_{\beta} M_{n-1} \rightarrow_{\beta} M_n \equiv N.]$$

(c)(Bonus exercise.) Prove that the reduction relation in (b) is m-complete for c.e. relations. Conclude that  $\twoheadrightarrow_{\beta}$  is not decidable. [Hint. Let  $P(n)$  be a c.e. relation, that is  $P(n) \iff \exists m. Q(n, m)$  for some computable (decidable) relation  $Q$ .

Step 1. Show that there exists an  $F \in \Lambda$  such that

$$F \mathbf{c}_n \mathbf{c}_k =_{\beta} \begin{cases} \lambda xy. x & \text{if } Q(n, k), \\ \lambda xy. y & \text{else.} \end{cases}$$

Step 2. Define  $H \triangleq WW$ , with

$$W \triangleq (\lambda wnk. Fnk(\lambda a_1 a_2 a_3 xy. x) wwn(\lambda bc. b(kbc))).$$

Show that  $\lambda bc. b(\mathbf{c}_k bc) \twoheadrightarrow_{\beta} \mathbf{c}_{k+1}$ .

Step 3. Show for all  $n, k \in \mathbb{N}$

- (i)  $Q(n, k) \Rightarrow H \mathbf{c}_n \mathbf{c}_k \twoheadrightarrow_{\beta} \lambda xy. x.$
- (ii)  $\neg Q(n, k) \Rightarrow H \mathbf{c}_n \mathbf{c}_k \twoheadrightarrow_{\beta} H \mathbf{c}_n \mathbf{c}_{k+1}.$
- (iii)  $P(n) \iff H \mathbf{c}_n \mathbf{c}_0 \twoheadrightarrow_{\beta} \lambda xy. x.$

Step 4. Finish the proof.]

## Answers

**Problem 1.** Suppose that such a primitive recursive  $F$  did exist. Then the function  $f(x) = F(x, x) + 1$  would also be primitive recursive. Hence we would have  $f(x) = F(n, x)$  for some  $n$  and every  $x$ . But then  $F(n, x) = F(x, x) + 1$  for every  $x$ , and by taking  $x = n$  we obtain a contradiction.

**Problem 2.** (a) Let  $d$  be the code of a p.c. function such that

$$\{d\}(x, n) = \begin{cases} 0 & \text{if } n = x \text{ or } \exists s\{x\}_s(n) \downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

Now define  $f(x) = S_1^1(d, x)$ . Then  $f$  is primitive recursive, hence total, and  $n \in W_{f(x)}$  iff  $n = x$  or  $n \in W_x$ .

(b) If  $W_x \subseteq \overline{K}$  and  $x \in W_x$ , then  $x \in K$  and also  $x \in \overline{K}$ , contradiction. Hence  $x \notin K$ , and hence also  $x \notin W_x$ .

(c) Let  $x$  be such that  $W_x = \emptyset$ . By (b) we have  $x \in \overline{K}$ . Now consider  $V = \{x, f(x), f^2(x), \dots\}$ . Then  $V$  is c.e., and by (b)  $V$  is infinite and  $V \subseteq \overline{K}$ .

**Problem 3.** (a)  $e \in C \Leftrightarrow \forall x \exists s \varphi_{e,s}(x) \downarrow = 0$ .

(b) Let  $A$  be an arbitrary  $\Pi_2^0$  set, say  $e \in A \Leftrightarrow \forall x \exists y R(e, x, y)$ , where  $R$  is computable. Let  $f$  be computable such that

$$\varphi_{f(e)}(x) = \begin{cases} 0 & \text{if } \exists y R(e, x, y), \\ \uparrow & \text{otherwise.} \end{cases}$$

Then  $e \in A \Leftrightarrow f(e) \in C$ .

**Problem 4.** (a)

The term  $\lambda xy. xyx$  has no type. If  $x$  (in the Church version) gets type  $A$ , then one should have  $A = B \rightarrow C$ , so that type  $C$  has less symbols than  $A$ . But  $C$  is the type of  $xy$  and this term should act on  $x$ , so we need  $C = A \rightarrow D$ , for some  $D$ , which is impossible as then type  $C$  has more symbols than  $A$ .

On the other hand  $\vdash \lambda xy. x(yx) : (\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \beta$ . Indeed,  $x:\alpha \rightarrow \beta, y:(\alpha \rightarrow \beta) \rightarrow \alpha \vdash yx : \alpha$ , hence

$x:\alpha \rightarrow \beta, y:(\alpha \rightarrow \beta) \rightarrow \alpha \vdash x(yx) : \beta$ .

Therefore the claim holds.

(b) One has  $\vdash \lambda xyz.y(xz) : (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)$ .

A derivation is:

$x:\alpha \rightarrow \beta, y:\beta \rightarrow \gamma, z:\alpha \vdash xz : \beta$

$x:\alpha \rightarrow \beta, y:\beta \rightarrow \gamma, z:\alpha \vdash y(xz) : \gamma$

$x:\alpha \rightarrow \beta, y:\beta \rightarrow \gamma \vdash \lambda z.y(xz) : \alpha \rightarrow \gamma$

$x:\alpha \rightarrow \beta, y:\beta \rightarrow \gamma \vdash \lambda yz.y(xz) : (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$

$\vdash \lambda xyz.y(xz) : (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$ .

Using inhabitation machines one sees that the type is not inhabited:

$$\boxed{\alpha \rightarrow \beta} \xrightarrow{\lambda f} \boxed{\beta} \longrightarrow x^\beta \quad \dots \overset{?}{\longrightarrow} \boxed{\alpha}$$

**Problem 5.** (a) The second fixed point theorem states that for all  $F \in \Lambda$  there exists an  $M \in \Lambda$  such that  $F \ulcorner M \urcorner =_\beta M$ . Apply this to  $F \equiv \text{l}$ . Then  $\ulcorner M \urcorner =_\beta \ulcorner \ulcorner M \urcorner \urcorner =_\beta M$ .

(b) Apply the second fixed point theorem to  $\text{Num}$  satisfying  $\text{Num} \ulcorner M \urcorner =_\beta \ulcorner \ulcorner M \urcorner \urcorner$ . Then

$$\ulcorner \ulcorner M \urcorner \urcorner =_\beta \text{Num} \ulcorner M \urcorner =_\beta M.$$

(c) We have  $\mathbf{c}_n =_\beta \mathbf{c}_m \Rightarrow n = m$  because the  $\mathbf{c}_n$  are in  $\beta$ -normal form, and a term can have at most one  $\beta$ -normal form. Now suppose

$$\ulcorner M \urcorner =_\beta \ulcorner \ulcorner M \urcorner \urcorner.$$

Then  $\mathbf{c}_{\#M} =_\beta \mathbf{c}_{\#\ulcorner M \urcorner}$ . Hence  $\#M = \#\ulcorner M \urcorner = \#\mathbf{c}_{\#M} > \#M$ , a contradiction.

**Problem 6.** (a)  $M \rightarrow_\beta N$  can be decided by observing the finitely many subexpressions of  $M$  of the form  $(\lambda x.P)Q$  and examining the result of contracting within  $M$  this subterm to  $P[x:=Q]$ : does it result in  $N$ ?

(b) Since a sequence of (codes of) terms can be coded as one number, part (a) implies that the relation is c.e..

(c) Given is a c.e. relation  $P(n) \iff \exists m \in \mathbb{N}. Q(n, m)$ , with  $Q$  decidable. Then  $\chi_Q(n, m)$ , the characteristic function of  $Q$ , is computable.

Step 1. By the  $\lambda$ -definability of computable functions there exists an  $F_Q \in \Lambda$  such that

$$F_Q \mathbf{c}_n \mathbf{c}_m = \begin{cases} \mathbf{c}_1 & \text{if } Q(n, m); \\ \mathbf{c}_0 & \text{else.} \end{cases}$$

Defining  $F \triangleq \lambda nm. F_Q nm (\mathbf{K}(\lambda xy.x)) (\lambda xy.y)$  one has

$$F \mathbf{c}_n \mathbf{c}_m = \begin{cases} \lambda xy.x & \text{if } Q(n, m); \\ \lambda xy.y & \text{else.} \end{cases}$$

Step 2. Following the hint we define

$$H \triangleq WW,$$

with

$$W \triangleq (\lambda wnk. Fnk(\lambda a_1 a_2 a_3 xy.x) wwn(\lambda bc.b(kbc))).$$

We note that

$$\lambda bc.b(\mathbf{c}_k bc) \equiv \lambda bc.b((\lambda fx.f^k x)bc) \rightarrow_\beta \lambda bc.b(b^k c) \equiv \lambda bc.b^{k+1} c \equiv \mathbf{c}_{k+1}.$$

Step 3(i). Suppose  $Q(n, k)$ . Then

$$\begin{aligned} H \mathbf{c}_n \mathbf{c}_k &\rightarrow_\beta WW \mathbf{c}_n \mathbf{c}_k \\ &\rightarrow_\beta F \mathbf{c}_n \mathbf{c}_k (\lambda a_1 a_2 a_3 xy.x) WW \mathbf{c}_n (\lambda bc.b(\mathbf{c}_k bc)) \\ &\rightarrow_\beta (\lambda xy.x) (\lambda a_1 a_2 a_3 xy.x) WW \mathbf{c}_n (\lambda bc.b(\mathbf{c}_k bc)) \\ &\rightarrow_\beta (\lambda a_1 a_2 a_3 xy.x) W \mathbf{c}_n (\lambda bc.b(\mathbf{c}_k bc)) \\ &\rightarrow_\beta (\lambda xy.x). \end{aligned}$$

(ii) Suppose  $\neg Q(n, k)$ . Then

$$\begin{aligned} H \mathbf{c}_n \mathbf{c}_k &\rightarrow_\beta WW \mathbf{c}_n \mathbf{c}_k \\ &\rightarrow_\beta F \mathbf{c}_n \mathbf{c}_k (\lambda a_1 a_2 a_3 xy.x) WW \mathbf{c}_n (\lambda bc.b(\mathbf{c}_k bc)) \\ &\rightarrow_\beta (\lambda xy.y) (\lambda a_1 a_2 a_3 xy.x) WW \mathbf{c}_n (\lambda bc.b(\mathbf{c}_k bc)) \\ &\rightarrow_\beta WW \mathbf{c}_n \mathbf{c}_{k+1} \\ &\equiv H \mathbf{c}_n \mathbf{c}_{k+1}. \end{aligned}$$

(iii) Suppose  $P(n)$ , that is  $Q(n, m)$  for some  $m$ , taken minimally. Then for  $k < m$  one has  $\neg Q(n, k)$ , so that

$$H \mathbf{c}_n \mathbf{c}_0 \rightarrow_\beta H \mathbf{c}_n \mathbf{c}_1 \rightarrow_\beta \cdots \rightarrow_\beta H \mathbf{c}_n \mathbf{c}_m \rightarrow_\beta \lambda xy.x.$$

Step 4. Therefore  $P$  is m-reducible to the relation  $\rightarrow_\beta$  on  $\lambda$ -terms. Since there are undecidable c.e. relations,  $\rightarrow_\beta$  cannot be decidable.