Exercises

For the part taught by Henk barendregt of Computability Theory, Mastermath Course, Fall 2014.

In red: correction found by some of the students or by me.

Notation $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$. The disjoint union of two sets $A, B \subseteq \mathbb{N}$ is denoted by $A \cup^* B$. The set of (closed) λ-terms is denoted by Λ (respectively $\Lambda^{\varnothing}$). Remember $K = \{x \mid \varphi_x(x)\downarrow\}$, $\mathsf{true} \triangleq \lambda xy.x$, $\mathsf{false} \triangleq \lambda xy.y$.

1. Week 6.10

1.1. Remember for $n, m \in \mathbb{N}_* = \mathbb{N} \cup \{*\}$ we write

$$\begin{array}{rcll} nm & = & \varphi_n(m) \;\; (= \varphi_n^{(1)}(m)) & \text{if defined;} \\ & = & * & \text{else, including } * \in \{n, m\}. \end{array}$$

We use association to the left.

(i) Show that there exists a $w \in \mathbb{N}$ satisfying for all $x, y \in \mathbb{N}_*$

$$wxy = xyy.$$

Solution. As xyy is partially computable in x, y there is an index e such that

$$\begin{array}{rcll} xyy & = & \varphi_e(x, y) & \\ & = & \varphi_{S(e,x)}(y), & \textit{applying the s-m-n theorem} \\ & = & \varphi_{\varphi_w(x)}(y), & \textit{idem.} \end{array}$$

(ii) Show that there exist $x, y \in \mathbb{N}$ such that $wxy = *$.

Solution. Let e be the index of the totally undefined function. Then

$$we0 \simeq e00 \simeq *0 = *.$$

(iii) Show that there exists a $k \in \mathbb{N}$ satisfying for all $x, y \in \mathbb{N}$

$$kxy = x.$$

Solution. Similar to (i).

(iv) Show that $\forall x, y \in \mathbb{N}_*.kxy = x$ doesn't hold.

Solution. $k0* = * \neq 0$.

1.2. Show that there exist $e, e' \in \mathbb{N}$ such that for all $x \in \mathbb{N}_*$

$$ex = e' + x \;\&\; e'x = e + 2x.$$

Solution. The double recursion theorem (1.11 of CT61014.pdf) states that for total computable functions f, g of two arguments there are e_1, e_2 such that

$$e_1 \sim f(e_1, e_2) \quad e_2 \sim g(e_1, e_2).$$

Now take $f(a, b)$ such that $\varphi_{f(a,b)}(x) = b + x$ (applying the S-m-n theorem to $\psi(a, b, x) = b + x$) and $g(a, b)$ such that $\varphi_{g(a,b)}(x) = a + 2x$. Now apply the double recursion theorem to obtain e, e' such that

$$e \sim f(e, e') \quad e' \sim g(e, e').$$

1.3. Show that there exists a computable function f such that the set of its fixed points

$$\mathcal{F}_f = \{e \in \mathbb{N} \mid f(e) \sim e\}$$

is not computable, by showing that $K \leq_m \mathcal{F}_f$. [Hint. This is possible for a relatively easy function f.]

Solution. For a certain computable function f we hope to construct a computable g such that for all $n \in \mathbb{N}$

$$\begin{aligned} n \in K \quad &\iff \quad f(g(n)) \sim g(n) \\ &\iff \quad \forall x. \varphi_{f(g(n))}(x) \simeq \varphi_{g(n)}(x). \end{aligned}$$

This goal is simplified by constructing f such that $\forall x, m. \varphi_{f(m)}(x) = 0$, by applying the S-m-n theorem to $\psi(m, x) = 0$. Then we want a total computable g such that

$$n \in K \quad \iff \quad \forall x. 0 = \varphi_{g(n)}(x).$$

Define

$$\chi(n, x) = \left\{ \begin{array}{ll} 0 & \textit{if } n \in K \\ \uparrow & \textit{else.} \end{array} \right\} = \varphi_{g(n)}(x),$$

by the S-m-n theorem. This g works.

2. Week 20.10

2.1. Write down a closed CL-term W consisting of $\mathsf{I}, \mathsf{K}, \mathsf{S}$ using applications such that (verify that it works!)

$$\mathsf{W}xy =_{\mathsf{CL}} xyy.$$

Solution. We know from theory that $\mathsf{W} \equiv [x]([y]xyy)$ *works:*

$$\mathsf{W}xy \equiv (([x]([y]xyy))x)y =_{\mathsf{CL}} ([y]xyy)y =_{\mathsf{CL}} xyy.$$

To write down this term we apply the algorithm (on p. 8 of CT201014.pdf): $[y]xyy = S([y]xy)([y]y) = S(S([y]x)([y]y))I = S(S(Kx)I)I$. *Hence* W *is*

$$\begin{aligned} [x]([y]xyy) &= [x]S(S(Kx)I)I \\ &= S(S(KS)(S(S(KS)(S(KK)I))(KI)))(KI). \end{aligned}$$

More efficiently (we use $[x]P = KP$ *and* $[x]Px = P$ *if* $x \notin \mathrm{FV}(P)$*)* $[y]xyy = S([y]xy)([y]y) = SxI$. *Hence* W *is*

$$\begin{aligned} [x]([y]xyy) &= [x]SxI \\ &= S([x]Sx)([x]I) \\ &= SS(KI). \end{aligned}$$

Verification: $SS(KI)xy = Sx(KIx)y = xy(Iy) = xyy$, *indeed!*

2.2. Write down an $F \in \Lambda^{\emptyset}$ such that (verify that it works!)

$$Fx =_{\beta} xF.$$

Solution. The desired equation follows from $F =_{\beta} \lambda x.xF$, *and this follows from* $F =_{\beta} (\lambda fx.xf)F$. *Thus we may take* F *as the fixed point of* $(\lambda fx.xf)$, *for example* $F \equiv \mathsf{Y}(\lambda fx.xf) =_{\beta} (\lambda zx.x(zz))(\lambda zx.x(zz))$. *Verification, writing* $D = (\lambda zx.x(zz))$:

$$Fx \equiv DDx \equiv (\lambda zx.x(zz))Dx \twoheadrightarrow_{\beta} x(DD) \equiv xF.$$

2.3. Let $\omega \triangleq \lambda x.xx$ and $1 \triangleq \lambda fx.fx$ $(\equiv \mathbf{c}_1)$. We may think that $\omega 1 =_{\beta} \mathsf{K}\omega$ (why?), but actually $\omega 1 =_{\beta} 1$. Show that from $\omega 1 = \mathsf{K}\omega$ one can derive any equation.

Solution. The following is seductive, but wrong:

$\omega 1 \to_{\beta} 11 \equiv (\lambda fx.fx)1 \to_{\beta} \lambda x.1x \equiv \lambda x.(\lambda fx.fx)x \to_{\beta} \lambda x.(\lambda x.xx) =_{\beta} \mathsf{K}\omega.$

The correct derivation from $\lambda x.1x$ *is:*

$\lambda x.1x \equiv \lambda x.(\lambda fx'.fx')x \to_{\beta} \lambda x.(\lambda x'.xx') \equiv \lambda xx'.xx' \equiv_{\alpha} \lambda fx.fx \equiv 1.$

For the derivation of a contradiction (any equation) note

$$\begin{aligned} \omega 1 = \mathsf{K}\omega \quad &\Rightarrow \quad \omega 1 ab = \mathsf{K}\omega ab \\ &\Rightarrow \quad 11ab = \omega b \\ &\Rightarrow \quad ab = bb \end{aligned}$$

Taking $a = KX$, $b = KY$, *we get* $KX(KY) = KY(KY)$, *hence* $X = Y$.

3. Week 27.10

3.1. Let $P = \{n \in \mathbb{N} \mid \exists p > n. p$ and $p + 2$ are primes$\}$. Let f be a computable function of two arguments. Define $Q = \{n \in \mathbb{N} \mid \neg \exists m. f(n,m) = m\}$.

(i) Show as warm-up that

$$\begin{aligned} K \cup \overline{K} &\leq_m K \cup^* \overline{K}; \\ K \cup^* \overline{K} &\not\leq_m K \cup \overline{K}. \end{aligned}$$

Solution. We have to show $\mathbb{N} \leq_m K \cup^* \overline{K}$, *and* $K \cup^* \overline{K} \not\leq_m \mathbb{N}$. *Let* $k \in K$*; then* $\langle k, 0 \rangle \in K \cup^* \overline{K}$. *Taking* $f(x) = \langle k, 0 \rangle$, *we have trivially* $f \colon \mathbb{N} \leq_m K \cup^* \overline{K}$. *For the inequality, note that we have* $k \notin \overline{K}$, *hence* $\langle k, 1 \rangle \notin K \cup^* \overline{K}$. *Suppose* $g \colon K \cup^* \overline{K} \leq_m \mathbb{N}$. *Then we should have* $g(\langle k, 1 \rangle) \notin \mathbb{N}$, *which is impossible. Therefore* $K \cup^* \overline{K} \not\leq_m \mathbb{N}$.

(ii) Now show

$$\begin{aligned} P &\leq_m K \cup^* \overline{K}; \\ Q &\leq_m K \cup^* \overline{K}. \end{aligned}$$

Solution. Note that P *is a ce set. Hence* $f \colon P \leq_m K$, *for some computable* f, *as* K *is ce-complete. Define* $f'(n) = \langle f(n), 0 \rangle$. *Then* $f' \colon P \leq_m K \cup^* \overline{K}$*:*

$$n \in P \iff f(n) \in K \iff f'(n) = \langle f(n), 0 \rangle \in K \cup^* \overline{K}.$$

Similarly Q *is co-ce, hence* $h \colon Q \leq_m \overline{K}$, *for* $h \colon \overline{Q} \leq_m K$. *Define* $h'(n) = \langle h(n), 1 \rangle$. *Then similarly* $h' \colon Q \leq_m K \cup^* \overline{K}$.

(iii) Show that in the future of mathematics it could be the case that $P \leq_m K \cup \overline{K}$.

Solution. It may be proved in the future that there are infinitely many prime twins. Then $P = \mathbb{N}$ *and trivially* $P \leq_m K \cup \overline{K}$.

(iv)* Show that already today $P \leq_m \{2n \mid n \in \mathbb{N}\}$, but not intuitionistically so!

Solution. By classical logic either there are infinitely many prime twins or not. In the first case $P = \mathbb{N}$ *and taking* $f(n) = 0$ *one has* $f \colon P \leq_m \{2n \mid n \in \mathbb{N}\}$. *In the second case* $P = \{0, \ldots, p\}$, *with* $p, p+2$ *the last prime twin. Define* $g(x) = 0$, *if* $x \leq p$, *else 1. Then* $g : P \leq_m \{2n \mid n \in \mathbb{N}\}$.
This reasoning uses the excluded middle and is not intuitionistic.

3.2. (i) Define the predicate

$$P(e,x) \triangleq \varphi_e \text{ is total and } \varphi_e(x) \sim x.$$

Show that $P \in \Pi^0_2$.

Solution. Note that $P(e,x) \iff$

$$\forall n \exists s. \varphi_{e,s}(n)\downarrow \ \&\ \forall m \exists s. [\varphi_{\varphi_{e,s}(x),s}(m) = \varphi_{x,s}(m)].$$

This is of the form $\forall\forall\exists\exists$*, hence in* Π^0_2*.*

(ii) What is the best position in the Arithmetical Hierarchy for P?

Solution. We claim that $\overline{K}_2 \leq_m P$*; then* P *is m-complete for* Π^0_2*, as* K_2 *is m-complete for* Σ^0_2*. It follows that* Π^0_2 *is the lowest level for* P *in the arithmetical hierarchy.*

To show the claim, remember $e \in \overline{K}_2 \iff \forall x \exists s. \varphi^2_{e,s}(e,x)\downarrow$*. Define*

$$\psi(e,x) = \left\{ \begin{array}{ll} x & \textit{if } \varphi_e(e,x)\downarrow \\ \uparrow & \textit{else} \end{array} \right\} = \varphi_{S(e)}(x),$$

by the S-m-n theorem. Then $\varphi_{S(e)}$ *is the identity if* $e \in \overline{K}_2$ *and always undefined otherwise. Therefore* $S\colon \overline{K} \leq_m P$*, since in case* $e \in \overline{K}_2$ *every* x *is a fixed point of* $\varphi_{S(e)}$*, being the identity.*

3.3. (i) Construct λ-defining terms for (see Syllabus CT) pd (predecessor), $\dot{-}$ (truncated subtraction), $\chi_{\geq}$.

Solution. Remember the defining schemes for these three functions:

$$\begin{array}{rcl} pd(0) & = & 0 \\ pd(x+1) & = & x \\ x \dot{-} 0 & = & x \\ x \dot{-} (y+1) & = & pd(x \dot{-} y) \\ \chi_{\geq}(x,y) & = & sg((x+1) \dot{-} y) \\ sg(0) & = & 0 \\ sg(n+1) & = & 1. \end{array}$$

The λ*-defining term for the successor is* $\texttt{suc} \triangleq \lambda nfx.f(nfx)$*.*

The λ*-defining terms for* $pd, \dot{-}, \chi_{\geq}$ *are* $\texttt{pred}, \dot{-}, F_{\geq}$ *respectively defined as follows.*

$$\begin{array}{rcl} \texttt{pred} & = & \lambda n.nT[0,0]\,\mathsf{false}, \qquad\qquad \textit{with } T = \lambda z.[\texttt{suc}(z\,\mathsf{true}), z\,\mathsf{true}]; \\ \texttt{monus} & = & \lambda xy.y\,\texttt{pred}\,x; \\ F_{\geq} & = & \lambda xy.\texttt{sg}(\texttt{monus}(\texttt{suc}\,x)y). \end{array}$$

(ii) Use the previous item and exercise 2 of CT271014.pdf to construct a λ-defining term of

$$g(n) = \mu x.[x + x \geq n].$$

Solution. First we construct a $B \in \Lambda^{\emptyset}$ such that

$$\begin{array}{rcl} B\mathbf{c}_0 & = & \mathsf{false}, \\ B\mathbf{c}_{k+1} & = & \mathsf{true}, \end{array}$$

taking $B \triangleq \lambda n.n(\mathsf{K}\,\mathsf{true})\,\mathsf{false}$. We want an $H \in \Lambda^{\emptyset}$ such that intuitively

$$\begin{array}{rcll} Hnx & = & x & \text{if } x + x \geq n \\ & = & Hn(x+1) & \text{else.} \end{array}$$

Then we can take $G \triangleq \lambda n.Hn\mathbf{c}_0$.
This H can be obtained by a fixed point construction satisfying

$$H =_\beta \lambda nx.B(F_{\geq}(A_{+}xx)n)x(Hn(\mathtt{suc}\,x)).$$

It suffices to take

$$H \triangleq \mathsf{Y}(\lambda hnx.B(F_{\geq}(A_{+}xx)n)x(hn(\mathtt{suc}\,x))).$$

4. Week 10.11

4.1. Let $\mathsf{Y} \triangleq \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$ and $\Theta \triangleq (\lambda ab.b(aab))(\lambda ab.b(aab))$. These terms are the fixed point combinators of Haskell Curry and Alan Turing, respectively. Show that

(i) $\mathsf{Y}f =_\beta f(\mathsf{Y}f)$ and $\Theta f =_\beta f(\Theta f)$.

Solution. Define $\omega_f \triangleq \lambda x.f(xx)$. Then

$$\mathsf{Y}f \equiv (\lambda f.\omega_f\omega_f)f =_\beta \omega_f\omega_f =_\beta f(\omega_f\omega_f) =_\beta f(\mathsf{Y}f).$$

(ii) $\mathsf{Y}f \not\twoheadrightarrow_\beta f(\mathsf{Y}f)$.
Solution. The reduction graph $G_\beta(\mathsf{Y}f)$ is:

$$\begin{tikzcd}
\mathsf{Y}f \equiv (\lambda f.f(\omega_f\omega_f))f \arrow[r, "\beta"'] \arrow[d, "\beta"'] & (\lambda f.f^2(\omega_f\omega_f))f \arrow[r, "\beta"'] \arrow[d, "\beta"'] & (\lambda f.f^3(\omega_f\omega_f))f \arrow[r, "\beta"'] \arrow[d, "\beta"'] & \cdots \\
f(\omega_f\omega_f) \arrow[r, "\beta"'] & f^2(\omega_f\omega_f) \arrow[r, "\beta"'] & f^3(\omega_f\omega_f) \arrow[r, "\beta"'] & \cdots
\end{tikzcd}$$

We see that $f(\mathsf{Y}f)$ never appears.

(iii) $\Theta f \twoheadrightarrow_\beta f(\Theta f)$.

Solution. Write $A \triangleq (\ ab.b(aab))$. *Then*

$$\Theta f \equiv AAF \to_\beta (\lambda b.b(AAb))f \to_\beta f(AAf) \equiv f(\Theta f).$$

(iv) There exists an $F \in \Lambda^\emptyset$ such that $Fx \twoheadrightarrow_\beta xF$.

Solution. $Fx \twoheadrightarrow_\beta xF$ *follows from* $F \twoheadrightarrow_\beta lx.xF$, *which follows from* $F \twoheadrightarrow_\beta (\lambda fx.xf)F$. *We can take* $F \triangleq \Theta(\lambda fx.xf)$ *and apply (iii).*

4.2. Define

$$\begin{array}{rcl} \mathcal{F}_1 & = & \{M \in \Lambda \mid \mathrm{FV}(M) = \{x\}\}, \\ \mathcal{F}_2 & = & \{M \in \Lambda \mid \exists N \in \Lambda . M =_\beta N \ \&\ \mathrm{FV}(N) = \{x\}\}. \end{array}$$

Which of these two sets is decidable (after coding)? Prove your answers.

Solution. $\mathcal{F}_1$ *is decidable, because when (a code of)* M *is given, then we can compute (a code of)* $\mathrm{FV}(M)$ *and see whether it is* $\{x\}$.
We have $\mathcal{F}_2 \neq \Lambda$: *one has* $y \notin \mathcal{F}_2$; *indeed, if* $y =_\beta N$, *then by the Church-Rosser theorm* $N \twoheadrightarrow_\beta y$ *and hence* $\mathrm{FV}(N) = \{x\}$ *is impossible, as free variables cannot be created during a reduction. Also* $\mathcal{F}_2 \neq \emptyset$: *one has* $x \in \mathcal{F}_2$. *Finally* $\mathcal{F}_2$ *is by definition closed under* $=_\beta$. *By Scott's theorem it follows that* $\mathcal{F}_2$ *is undecidable.*

4.3. Show that there exists a term $F \in \Lambda^\emptyset$ such that for all $G \in \Lambda$ one has

$$F\ulcorner G \urcorner = \begin{cases} \mathsf{true} & \text{if } G \equiv F \\ \mathsf{false} & \text{else} \end{cases}$$

[Hint. F may be called a *selfish* term. Use that $\equiv$ (up to α-equivalence) is decidable, that computable functions are λ-definable, and the second fixed point theorem.]

Solution. Following the hint there exists an $H \in \Lambda^\emptyset$ *such that*

$$H\ulcorner F \urcorner \ulcorner G \urcorner = \begin{cases} \mathbf{c}_0, & \text{if } F \equiv G \\ \mathbf{c}_1, & \text{else.} \end{cases}$$

We can modify H *to* $H' \triangleq \lambda fg.Hfg(\mathsf{K}\mathsf{true})\,\mathsf{false}$ *such that*

$$H'\ulcorner F \urcorner \ulcorner G \urcorner = \begin{cases} \mathsf{true}, & \text{if } F \equiv G \\ \mathsf{false}, & \text{else.} \end{cases}$$

By the second fixed point theorem there exists an $F \in \Lambda$ *such that*

$$H'\ulcorner F \urcorner =_\beta F.$$

Then $\mathrm{FV}(F) = \mathrm{FV}(H') = \emptyset$, *so* $F \in \Lambda^\emptyset$, *and has the required property*

$$F\ulcorner G \urcorner = H'\ulcorner F \urcorner \ulcorner G \urcorner \begin{cases} \mathsf{true}, & \text{if } F \equiv G \\ \mathsf{false}, & \text{else.} \end{cases} .$$

5. Week 17.11

5.1. Show that there is a term $D{\in}\Lambda^{\emptyset}$ such that for all $M{\in}\Lambda^{\emptyset}$

$$D\ulcorner M\urcorner =_\beta M\ulcorner\ulcorner M\urcorner\urcorner.$$

Solution. Note that $\mathsf{E}\ulcorner M\urcorner =_\beta M$ *and* $\mathsf{Num}\ulcorner M\urcorner =_\beta \ulcorner\ulcorner M\urcorner\urcorner$ *for all* $M{\in}\Lambda^{\emptyset}$. *Hence we can take* $D \triangleq \lambda m.\mathsf{E}m(\mathsf{Num}\ m)$.

5.2. (i) Find a term $M{\in}\Lambda$ such that its reduction graph $G_\beta(M)$ looks like

$$\begin{tikzcd} M \arrow[r, bend right] & \cdot \arrow[l, bend right] \end{tikzcd} \qquad \text{(ii) Idem for} \quad \begin{tikzcd} M \arrow[rr, bend right] \arrow[dr] & & \cdot \arrow[ll, bend right] \arrow[dl] \\ & \cdot & \end{tikzcd}$$

Solution. (i) We try $M \equiv AB$, *with* $A \equiv \lambda a.a\cdots$, $B \equiv \lambda b.C[b]$. *Then* $M \equiv AB \to_\beta B\cdots \to_\beta C[\cdots]$. *If the latter is going to be* AB, *then choosing* $A \triangleq \lambda a.a\mathsf{I}a$, $B \triangleq \lambda ba.aba$ *works:* $AB \to_\beta B\mathsf{I}B \to_\beta (\lambda a.a\mathsf{I}a)B$. *Simpler:* $A \triangleq \lambda a.aaa$, $B \triangleq \lambda x.A$. *Then*

$$AB \to_\beta BBB \equiv (\lambda x.A)BB \to_\beta AB.$$

(ii) Take $M \triangleq (\lambda x.\mathsf{I})(AB)$, *with* AB *as in (i). With the second solution of (i) one can take* $M \triangleq B(AB)$.

5.3. Find (simple) types for the following λ-terms: (i) $\lambda xy.xy(xyy)$.
(ii) $\lambda xy.x(yx)$. (iii) $\lambda xy.x(yxx)$.

Solution.
(i)

$$\cfrac{\cfrac{\cfrac{x{:}\alpha^2{\to}\alpha, y{:}\alpha \vdash xy : \alpha{\to}\alpha \qquad x{:}\alpha^2{\to}\alpha, y{:}\alpha \vdash xyy : \alpha}{x{:}\alpha^2{\to}\alpha, y{:}\alpha \vdash xy(xyy) : \alpha}}{x{:}\alpha^2{\to}\alpha \vdash \lambda y.xy(xyy) : \alpha{\to}\alpha}}{\vdash \lambda xy.xy(xyy) : (\alpha^2{\to}\alpha){\to}\alpha{\to}\alpha}$$

This solution is not complete: the 'axiom'

$$x{:}\alpha^2{\to}\alpha, y{:}\alpha \vdash xy : \alpha{\to}\alpha$$

really should have been derived

$$\cfrac{x{:}\alpha^2{\to}\alpha, y{:}\alpha \vdash x : \alpha{\to}\alpha{\to}\alpha \qquad x{:}\alpha^2{\to}\alpha, y{:}\alpha \vdash y : \alpha}{x{:}\alpha^2{\to}\alpha, y{:}\alpha \vdash xy : \alpha{\to}\alpha},$$

and similarly for the other 'axiom' $x{:}\alpha^2{\to}\alpha, y{:}\alpha \vdash xyy : \alpha$.

(ii)

$$\cfrac{\cfrac{\cfrac{x{:}\alpha{\to}\beta, y{:}(\alpha{\to}\beta){\to}\alpha \vdash x : \alpha{\to}\beta \qquad x{:}\alpha{\to}\beta, y{:}(\alpha{\to}\beta){\to}\alpha \vdash yx : \alpha}{x{:}\alpha{\to}\beta, y{:}(\alpha{\to}\beta){\to}\alpha \vdash x(yx) : \beta}}{x{:}\alpha{\to}\beta \vdash \lambda y.x(yx) : ((\alpha{\to}\beta){\to}\alpha){\to}\beta}}{\vdash \lambda xy.x(yx) : (\alpha{\to}\beta){\to}((\alpha{\to}\beta){\to}\alpha){\to}\beta}$$

(iii) *In this item we show how the types are found, working 'bottom up'. We want*

$$\frac{}{\vdash \lambda xy.x(yxx) : 1}$$

This can come only from

$$\frac{x{:}2 \vdash \lambda y.x(yxx) : 3}{\vdash \lambda xy.x(yxx) : 1 = 2{\to}3}$$

And this only from

$$\frac{\dfrac{x{:}2, y{:}4 \vdash x(yxx) : 5}{x{:}2 \vdash \lambda y.x(yxx) : 3 = 4{\to}5}}{\vdash \lambda xy.x(yxx) : 1 = 2{\to}3}$$

For this we need $4 = 2{\to}2{\to}6$ *and* $2 = 6{\to}5$, *obtaining*

$$\frac{\dfrac{x{:}(6{\to}5), y{:}(6{\to}5)^2{\to}6 \vdash x(yxx) : 5}{x{:}(6{\to}5) \vdash \lambda y.x(yxx) : ((6{\to}5)^2{\to}6){\to}5}}{\vdash \lambda xy.x(yxx) : (6{\to}5){\to}((6{\to}5)^2{\to}6){\to}5}$$

Or with a renaming

$$\frac{\dfrac{x{:}(\alpha{\to}\beta), y{:}(\alpha{\to}\beta)^2{\to}\beta \vdash x(yxx) : \alpha}{x{:}(\alpha{\to}\beta) \vdash \lambda y.x(yxx) : ((\alpha{\to}\beta)^2{\to}\alpha){\to}\beta}}{\vdash \lambda xy.x(yxx) : (\alpha{\to}\beta){\to}((\alpha{\to}\beta)^2{\to}\alpha){\to}\beta}$$

6. Week 01.12

6.1. Let $o{\in}\mathbb{T}^{\mathbb{A}}$. Remember that $0 = o$, $n+1 = n{\to}o$. Define

$$\Lambda(A) = \{M{\in}\Lambda^{\emptyset} \mid \ \vdash_{\lambda^{\mathrm{cu}}_{\to}} M : A\}.$$

We will show that for $A{\in}\{0,\ 2,\ ,4,\cdots\}$ one has $\Lambda(A) = \emptyset$.

(i) Show $\Lambda(0) = \emptyset$. [Hint. Suppose that $\vdash_{\lambda^{\mathrm{cu}}_{\to}} M : 0$, with $M{\in}\Lambda^{\emptyset}$. We may assume that M is in β-nf (why?). If $M \equiv PQ$, then M is not in nf. If $\vdash M \equiv \lambda x.P : A$, then $A \equiv B{\to}C$. If $M \equiv x$, then $M \notin \Lambda^{\emptyset}$.]

Solution. Suppose $M{\in}\Lambda(0)$. *By the normalization theorem* M *has a normal form* N. *By the Church-Rosser theorem* $M \twoheadrightarrow_\beta N$. *Since typing is preserved under* β*-reduction one has* $N{\in}\Lambda(0)$. *Every (untyped) lambda term is of the form* $\lambda\vec{x}.y\vec{R}$ *or* $\lambda\vec{x}.(\lambda y.P)Q\vec{R}$. *Since* 0 *is an atomic type for* $N{\in}\Lambda(0)$ *one has* $\vec{x} = \emptyset$*: this means that* N *is either of the form* $y\vec{R}$ *or* $(\lambda y.P)Q\vec{R}$. *The first case is not possible as* N *is closed, the second not as* N *is in normal form.*

(ii) Show $\Lambda(2) = \emptyset$. [Hint. use that $\mathsf{I}{\in}\Lambda(1)$ and (i).]

Solution. If $M{\in}\Lambda(2)$, then $M\mathsf{I}{\in}\Lambda(0)$, which is not possible by (i).

(iii) Show $\Lambda(3) \neq \emptyset$. [Hint. Consider $M \equiv \lambda F^2.F^2\mathsf{I})$.]

(iv) Show $\Lambda(4) = \emptyset$. *Similar to (ii), using (iii).*

(v) Show $\Lambda(2n-1) \neq \emptyset,\ \Lambda(2n) = \emptyset$, for all $n > 0$.

Solution. We show $\Lambda(5) \neq \emptyset$ from which follows as before that $\Lambda(6) = \emptyset$. An inhabitant $M_5{\in}\Lambda(5)$ should be of the form $M_5 \equiv \lambda F^4.N$, with N of type 0. We can find such an N by taking $N \equiv F^4 M_3$, with $M_3{\in}\Lambda(3)$, according to (iii). The general argument is by induction.

6.2. We study ways in which the proof of SN for $\lambda_{\rightarrow}^{\mathrm{CH}}$ after some modification doesn't hold.

(i) If one doesn't add constants, where does the proof break down?

Solution. We need terms in $\mathcal{C}$ in order to be able to substitute and give arguments to a term in $\mathcal{C}^*$; this is needed in the proof of (4) on page 7/11 in CT011214.pdf.

(ii) If one defines $\mathcal{C}^*_A \triangleq \mathcal{C}_A$, where does the proof break down?

Solution. In the proof of (5), in the same proof, step (7) case $M \equiv (\lambda x.P)$ would fail. In that step we want to show that $(\lambda x.P){\in}\mathcal{C}^*$, knowing that $P{\in}\mathcal{C}^*$. The argument $Q{\in}\mathcal{C}$ given to $(\lambda x.P)$ gets swallowed by P resulting in $P[x{:=}Q]$ and the induction hypothesis ($P{\in}\mathcal{C}^*$) needs to deal with substituion results.

(iii) If one defines $\mathcal{C}_A \triangleq \{M{\in}\Lambda(A) \mid M{\in}\mathrm{SN}\}$, where does the proof break down?

Solution. Now in the case $M \equiv PQ$ of step (7) in the same proof we (perhaps) don't have that $P, Q{\in}\mathrm{SN} \Rightarrow (PQ){\in}\mathrm{SN}$.

6.3. We will show that $\vdash_{\lambda_{\rightarrow}^{\mathrm{cu}}} N : A\ \&\ M \twoheadrightarrow_\beta N \not\Rightarrow \vdash_{\lambda_{\rightarrow}^{\mathrm{cu}}} M : A$.

(i) Show that $\mathsf{SK} \twoheadrightarrow_\beta \mathsf{false}$.

Solution.

$$\mathsf{SK} \equiv (\lambda abc.ac(bc))\mathsf{K} \rightarrow_\beta \lambda bc.\mathsf{K}c(bc) \rightarrow_\beta \lambda bc.c \equiv \mathsf{false}.$$

(ii) Show $\vdash \mathsf{false} : \alpha{\rightarrow}\beta{\rightarrow}\beta$. *Solution.*

$$\dfrac{\dfrac{x{:}\alpha, y{:}\beta \vdash y : \beta}{x{:}a \vdash \lambda y.y : \beta{\rightarrow}\beta}}{\vdash \lambda xy.y : \alpha{\rightarrow}\beta{\rightarrow}\beta}$$

(iii) Show $\vdash \mathsf{SK} : (\beta{\to}\gamma){\to}\beta{\to}\beta$.

Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{x{:}\alpha{\to}\beta{\to}\gamma, y{:}\alpha{\to}\beta, z{:}\alpha \vdash xz(yz) : \gamma}{x{:}\alpha{\to}\beta{\to}\gamma, y{:}\alpha{\to}\beta \vdash \lambda z.xz(yz) : \alpha{\to}\gamma}}{x{:}\alpha{\to}\beta{\to}\gamma \vdash \lambda yz.xz(yz) : (\alpha{\to}\beta){\to}\alpha{\to}\gamma}}{\vdash \mathsf{S} \equiv \lambda xyz.xz(yz) : (\alpha{\to}\beta{\to}\gamma){\to}(\alpha{\to}\beta){\to}\alpha{\to}\gamma}}{} \qquad \dfrac{\dfrac{x{:}\alpha, y{:}\beta \vdash x : \alpha}{x{:}a \vdash \lambda y.x : \beta{\to}\alpha}}{\vdash \mathsf{K} \equiv \lambda xy.x : \alpha{\to}\beta{\to}\alpha}$$

In order to fit K *as argument for* S *we must unify the types by substitution* $[\gamma{:}=\alpha]$*:*

$$\dfrac{\dfrac{\dfrac{\dfrac{x{:}\alpha{\to}\beta{\to}\alpha, y{:}\alpha{\to}\beta, z{:}\alpha \vdash xz(yz) : \alpha}{x{:}\alpha{\to}\beta{\to}\alpha, y{:}\alpha{\to}\beta \vdash \lambda z.xz(yz) : \alpha{\to}\alpha}}{x{:}\alpha{\to}\beta{\to}\alpha \vdash \lambda yz.xz(yz) : (\alpha{\to}\beta){\to}\alpha{\to}\alpha}}{\vdash \mathsf{S} \equiv \lambda xyz.xz(yz) : (\alpha{\to}\beta{\to}\alpha){\to}(\alpha{\to}\beta){\to}\alpha{\to}\alpha} \qquad \dfrac{\dfrac{x{:}\alpha, y{:}\beta \vdash x : \alpha}{x{:}a \vdash \lambda y.x : \beta{\to}\alpha}}{\vdash \mathsf{K} \equiv \lambda xy.x : \alpha{\to}\beta{\to}\alpha}}{\vdash \mathsf{SK} : (\alpha{\to}\beta){\to}\alpha{\to}\alpha}$$

This type for SK *is a renaming variant of* $(\beta{\to}\gamma){\to}\beta{\to}\beta$*.*

(iv) Show $\nvdash \mathsf{SK} : \alpha{\to}\beta{\to}\beta$.

Solution. The derivation under (iii) of a type for SK *shows it is minimal.*