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KEYS TO TWO INTIMACIES
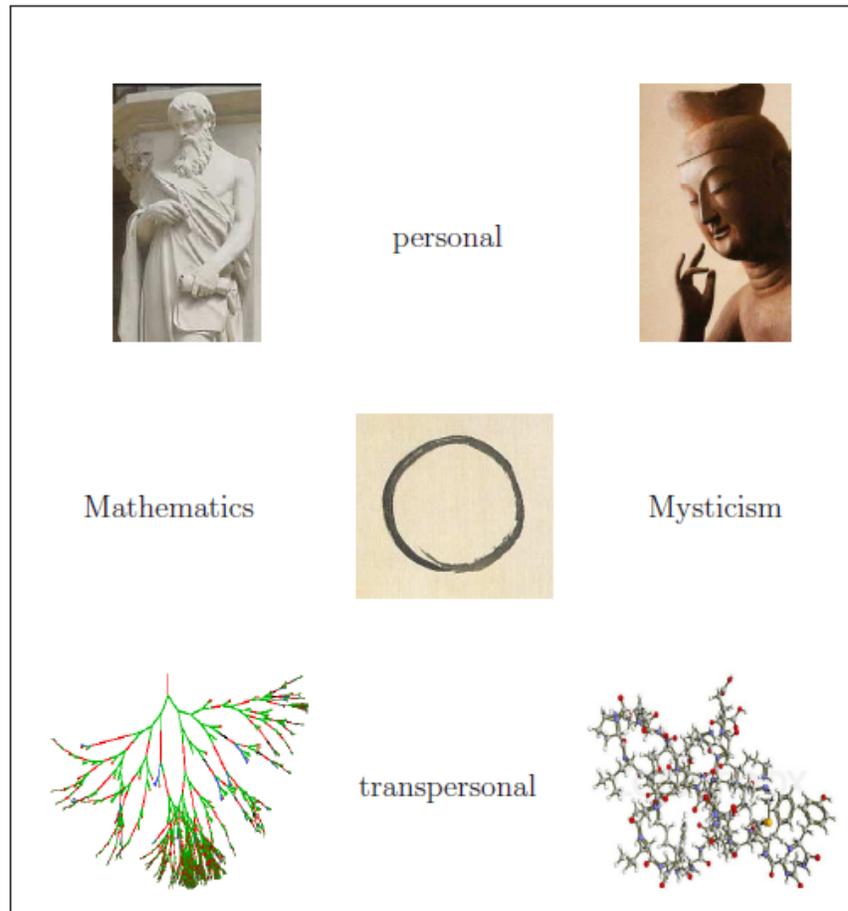
Mathesis and Mysticism[1]

Fig. 1. From left to right and from top to bottom:
Euclid, Buddha, representation of emptiness,
formal proof that there are infinitely many primes, β-endorphin.

SUMMARY – The subjects mathematics and mysticism come from my main scientific interests: logic and meditation phenomena. We will discuss two types of keys for these: the personal, through inner experience, and the transpersonal, described as an objective natural process. The two types of keys exist for both subjects.

In the novel Dr. Faustus of Thomas Mann two friends speak about human emotions and experiencing the divine in music. 'One should love them', said the first. 'Do you believe love is the strongest emotion?' asked the other. 'Do you know any stronger?' asked the former. 'Yes, interest'.

This passage shows that love and interest compete for being considered as the strongest emotion. This is sometimes described by romantics as the opposition between heart and intelligence or also between body and mind. But for most emotions a coordinated cooperation between body and mind is important.

The purest form of intelligence focuses on mathematics. The highest form of love is seen as the mystic unity with God. But to explain the mystical experience the hypothesis of the existence of the Supreme Being is not necessary. One can also understand it, like in Buddhist psychology, as a state of mind with a high form of concentration.

First there are the private keys to the two subjects. The experience of the mathematical truth through the mental activity of proving. This requires curiosity and study. The mystical experience is achieved through the practice of meditation, using the similar tools of motivation and commitment. Both for experiencing mathematics and mysticism one only can create the right conditions, the rest is--one could say--divine grace. This is how it is described in many traditions of mysticism. But also among mathematicians this parlance is in vogue. For example the Polish logician Andrzej Mostowski spoke with admiration about his American colleague Robert Solovay: 'He must have a direct phone line to God.' So far the personal keys to mentioned subjects.

Then there are the transpersonal keys, apparently destroying the soul from both subjects. But I believe that the topics do gain depth through the combination of the personal and transpersonal aspects.

1. TWO KEYS TO MATHEMATICS

1.1 Personal

In his novel *The Man Without Qualities*, Robert Musil wrote the following about the main character, a mathematician, who contemplates as follows about his profession:

> *The precision, strength and certainty of this thinking, nowhere equaled in life, overwhelmed him almost with melancholy.*[2]

We will give a simple example of this mathematical thinking.

1.1.1 Definition. A positive integer is called a *prime* (number) if p>1 and p has only 1 and itself as divisor.

For example, of the numbers below ten 2, 3, 5 and 7 are prime; not prime are 1 (by definition), 4, 6, 8 and 9. We do not want that 1 is prime for aesthetic reasons, related to the following fact, which we state without proof.

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2 *Die Genauigkeit, Kraft und Sicherkeit dieses Denkens, die nirgends im Leben ihresgleichen hat, erfühlte ihm Fast mit Schwermut*. R. Musil, *Der Mann ohne Eigenschaften*. Erstes Buch, Berlin: Rowohlt, 1930, kapitel 29.

1.1.2 Proposition. *Each positive integer can be written as the product of a unique sequence (apart from the order, and possibly with duplicates or empty; 1 can be considered as the product of the empty sequence) of primes (prime decomposition). Therefore each number >1 has a prime divisor, i.e. a divisor that is prime.*

For example 12=2 x 2 x 3. If one considers 1 as a prime, then also 12=1 x 2 x 2 x 3 is a prime decomposition, which is therefore no longer unique.

The ancient Greeks knew the primes. They could not only enumerate them, but also ask and answer questions about these. Consider the sequence of primes:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, …

Does this sequence ever end or does it continue indefinitely?

1.1.3 Theorem (Euclid). *There are infinitely many primes.*[3]

Proof. Consider a finite list of primes p_1, … , p_n. Define the product $P = p_1$ x p_2 x … x p_n. Let q be a prime factor of P+1. Clearly p_1>1 divides P, so it doesn't divide P+1. Now q by definition divides P+1, therefore $q \neq p_1$. Similarly $q \neq p_2$, … , $q \neq p_n$. Hence q is a prime number that doesn't appear in the original list. We see that every finite list of primes can be extended with a new one. Conclusion: there are infinitely many primes. QED

A small variation of this proof shows that for each number n a prime number can be found that is at most n!+1 = (1 x 2 x $\cdots$ x n)+1. In the for mathematics very rich 19th century it was proved that the first prime number greater than n>1 is at most 2n.

We can't control when mathematical insight arrives. The French mathematician Poincaré (1908) wrote that, when he was drafted for military service, he gave up to find a solution to a particular problem, after having spent a long time on it. After his military service he stepped on a good day in a tram. Placing his foot he suddenly saw a possible solution for his problem, which came to him unexpectedly via direct intuition. This still had to be verified rationally; at home it turned out to be correct. Poincaré conjectured that his intuition came from subliminal thinking. Later his colleague and countryman Hadamard (1945) extended the conjecture by stating that this subconscious intuition probably utilizes parallel processing.

Although mathematics is generally considered to be the most precise of all sciences, it is less known that the final verdict rests on a judgment based entirely on 'inner vision'. Despite the reasoning and calculations like in the proof above, it still needs the inner views to see the correctness of the reasoning, and the applicability of the calculation. This was true at least until the end of the 20th century. The situation now slowly starts to change.

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3 The result of Euclid on the infinity of the collection of primes immediately leads to other questions. A *prime-twin* is a pair of prime numbers with 2 as difference. For example, (3,5) and (5,7) are prime-twins, but the pair (7.9) is not. An unanswered conjecture is whether there exist infinitely many prime-twins. This simple question gives rise to complex mathematics. If one requires for two primes that they may differ up to 600, then there are infinitely many such couples, J. Maynard (2013) (arxiv.org/abs/1311.4600), where the number 600 an improvement of 70 million, previously found by Y. Zhang. In the meantime, the result appears to be improved in a joint project Polymath8 (arxiv.org/abs/1409.8361) to 246 as maximum difference. By this one hasn't yet arrived at the difference 2 establishing the prime-twin conjecture; moreover the used methods probably can't reach it.

1.2 Transpersonal

For the transpersonal keys to mathematics two fundamental ideas of the Greek philosopher Aristotle (Posterior analytics) are of interest. His first contribution was the following description of the *axiomatic method* in mathematics. On the one hand there are mathematical *objects* (also known as *concepts* as these consist mainly in the mind), for example, numbers such as 2, ¾ and √5, or geometric figures such as triangles, ellipses and pyramids. On the other hand, there are *qualities* of those objects. It is worthwhile to find out what are the valid qualities, the so called theorems, that hold for an object or class of objects. (Example of a not valid quality: 'The number 6 is a prime'.)

How does one arrive at mathematical objects and theorems concerning these? According to Aristotle objects are obtained from previously found objects by means of a *definition*. Theorems one finds from previously obtained ones by a *proof*. In order to avoid an infinite regression, one has to assume *primitive concepts*, that do not get a definition. Similarly, there are *axioms*, theorems that do not get a proof. On the basis of the primitive notions and axioms, using definitions and proofs, one can develop a mathematical theory. The axiomatic method was considered in ancient Greece as follows: primitive concepts are so clear that they don't need a definition; similarly the axioms are so true that no proof is needed for them. This view is not completely satisfactory. More than 2000 years later the German mathematician Hilbert gave an elegant interpretation of the axiomatic method. He considered the axioms as an implicit definition of the primitive concepts. He did not mind what is the essence of a point or a line, as long as there is exactly one line through two distinct points, one of the axioms of the Euclidean geometry, and also the other axioms hold.

Proofs are based on intuitive reasoning. A second important contribution of Aristotle for the foundations of mathematics was his project to make a map, called *logic*, of all possible ways of reasoning. This enterprise was completed only about 2300 years later by a proposal of Frege (1879), of which Gödel (1930) showed that it was right indeed. From that moment on proofs, and hence the consequences of an axiomatic system, were completely determined: a proof is a sequence of statements, which either are guaranteed by the axioms or result from previous statements on the basis of the rules of logic.

In this way, it is possible to write down proofs in a complete formal way and it is easy to determine objectively its validity: step by step. Then Frege (1893) began to formalize a (modest) part of arithmetic. Unfortunately, the logician Bertrand Russell found that a contradiction could be derived from the axioms used by Frege. This made the latter propose to withdraw his work, but his publisher wanted to publish it anyway. Thereafter Russell collaborated with the mathematician Whitehead to formalize a portion of mathematics (including arithmetic) within an axiom system considered to be safe. This resulted in the first part of the monumental *Principia Mathematica*, Whitehead and Russell (1910). This work, however, had the disadvantage that it is virtually unreadable; moreover Principia Mathematica was also formulated less precisely than the work of Frege. This being so, who then verifies the formal proofs of Principia Mathematica?

This question was answered by the Dutch mathematician N.G. de Bruijn (1970), who showed how a computer could efficiently verify a formalization in a certain formalism. The required program is very small, essentially a good description of the rules of the logic, which De Bruijn had displayed in a more convenient way than Frege.

However, most mathematicians stayed away from formalizing: it was difficult, incomprehensible to humans, and the verified theorems were not advanced. In the beginning of the 90s of the last century it was still laborious to prove formally the following statements:

- 17 is a prime;
- $(x+1)(x-1)=x^2-1$ holds for all real numbers;
- $(e^x+e^{-x})/2$ is a continuously differentiable function in the real variable x.

For depth in formal mathematics new ideas were needed. These came from unexpected sources: (i) computer algebra systems; (ii) the method of proof of the incompleteness theorem of Gödel.

As to (i). Computer Algebra Systems are powerful, large, but therefore not always reliable. By using them in a skeptic way, similar to the role of intuition, which is later to be verified, entirely reliable results can be obtained. This was presented in Barendregt and Cohen (2001). This technique eventually led to the certification of primes of hundreds of digits: first by Caprotti and Oostdijk (2001) for numbers with up to 100 digits, generating formal proof of their primality on the basis of the little theorem Fermat (Pocklington criterion); thereafter, the method was extended to numbers with up to 300 digits on the basis of a decision method that uses elliptic curves, Théry and Henrot (2007).

As to (ii). Being one of the few who had read and understood *Principia Mathematica*, Gödel (1931) has shown that there are statements, what can neither be proved nor be refuted (unless used axioms lead to inconsistencies, such as the one used by Frege). This so-called incompleteness theorem seems to plea against formalization. The meaning is, however, that the axiomatic method has its limitation, regardless of whether proofs are formal or intuitive, but the axiomatic method is very strong, despite the incompleteness theorem. Moreover, we have nothing better: there is no algorithm that determines whether a statement is valid or not. This was proved by Turing (1937) with as spin off the design of the programmable computer, which most of you carry around in the form of a smart phone with apps. Gödel demonstrated his incompleteness result by encoding mathematical concepts as numbers, so that arithmetic can indirectly make statements about itself. The application of the laws of logic is then a calculable operation on these numbers, which is representable within axiomatic arithmetic. This so-called 'reflection' was utilized in Barendregt and Barendsen (2002) to generate formal proofs on the basis of the syntactical properties of the statements. This provides e.g. formal correctness proofs for arbitrary multiplication of polynomials. In Cruz-Filipe (2003) reflection is used to generate formal evidence for continuity, differentiation and the calculation of the derivative of well-known real functions.

Formal proofs can be seen as keys opening boxes containing fully developed mathematical understanding; the latter can be utilized for the construction of new and more complex keys. There are also 'dynamic keys', which unfold during their use to the correct shape. These correspond to proofs using calculations. The HOL system (<en.wikipedia.org/wiki/HOL_(proof_assistant)>) for formal mathematics employs dynamic keys that first must be unfolded, before verification is possible. In other systems, such as Coq (<coq.inria.fr>), it is not necessary to perform the unfolding beforehand, because these systems also contain a computational model for a programming language, and one may provably show in advance that an expanded key

will do its required work. This then can be called an *efficiently dynamic proof key*[4]. One of the first extensions of a logical system with an external computing device is given in Gödel (1958), where an operator for higher-order primitive recursion is introduced.

Reflection together with efficient expandable proof keys form one of the basic ingredients used for the research in Nijmegen, obtaining formal proofs, for among others mathematical analysis.[5] A formal proof of the Theorem of Euclid that there are infinitely many primes is displayed in Fig. 2.
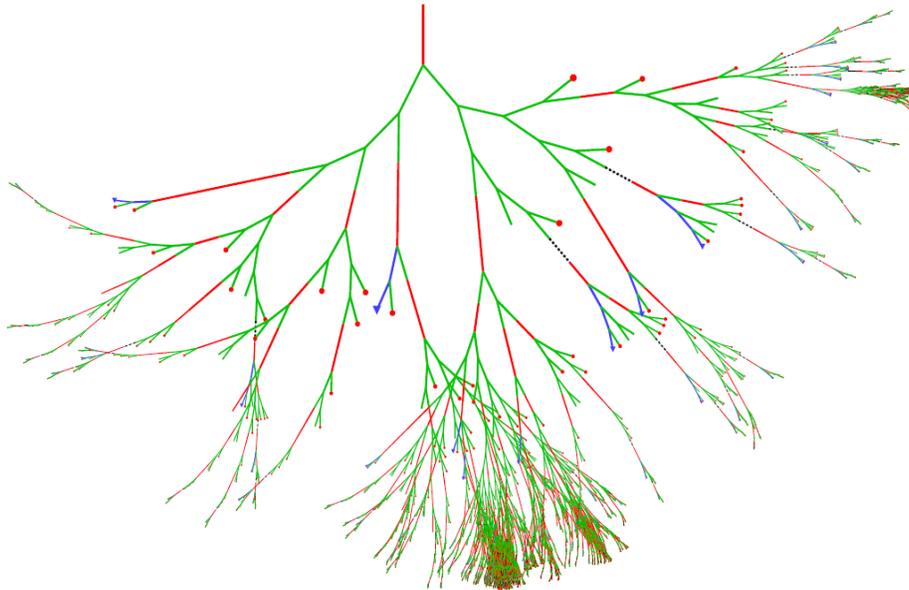

Fig. 2. Formal proof that there exist infinitely many primes. Formalization: Freek Wiedijk; image: Joerg Endrullis. It is a tree with approximately 71000 nodes (forkings and end-points). This seems too much: the intuitive evidence is short and the formal is long. We should be recalled that for the realization of an easy operation, such as touching the tip of our nose with the index finger, many neurons and synapses will be involved. Many of the intuitively clear steps in the proof are completely spelled out.

Georges Gonthier (2008, 2014) explains that the method of reflection is used as an important tool for his impressive formalizations of the Four color theorem in combinatorics and the theorem of Feit-Thompson (1963) from group theory. By Hales et al. (2015) another tour de force has been performed, both from the intellectual and organizational point of view.[6] This consists of the formalization of Hales' proof of the Kepler conjecture about the most efficient space packing of spheres (like oranges are

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4 There are in Coq parameterized proofs p(x) for which it can be shown for all natural numbers n that the expression p(n) is evidence of a certain quality E(n) without needing to expand p(n). In Bruijn (1970) the lambda calculus is extended with new typing rules such that such dynamical evidence can be represented.

5 Constructive proofs of the fundamental theorem of algebra (Geuvers et al. [2002]) and analysis (Cruz-Filipe [2003]). In O'Connor [2008] certified proofs are given for the correctness of efficient algorithms for infinitely precise calculations with real Numbers.

stacked in an old-fashioned vegetable store). The Foundations Department at the Faculty of Science of Radboud University can be proud that Hales has spent a sabbatical in Nijmegen and that two of its former collaborators became co-author of the groundbreaking article. It can rightly be said that formalizing of proofs is possible for 'non-trivial' (an understatement) results. However, at present still with much effort: according to an estimate of Freek Wiedijk the needed time to construct a formalization of a proof is on the average about ten times as long as writing it down informally. This needs to be improved.

1.3 A Comparison

The first reaction on formalized proofs by many mathematicians was disapproval. Because usually proofs are developed in their mind, one considered formal proofs, which are usually too large to be overlooked entirely, as a treason against the spirit of mathematics. Such an attitude undoubtedly also must have existed in the early days of cellular biology. While in the old days a practitioner of 'natural history' romantically went into the fields to catch plants and butterflies as object of study, later biological research was carried out by using microscopes to observe cells.

Because of this reluctance to formalize, the automated verification of proofs in mathematics was preceded by that of correctness proofs of first hardware and later software. For this, interactive systems were built, the so-called *proof assistants*, which are helpful in construction of formal proofs within IT applications.[7] These applications have been instrumental for the rise of formal mathematics.

My goal during the occupation of the chair of Foundations was to make more familiar the act of providing formal proofs in mathematics. It was my expectation that by steadily working on the formalization of the Master's topics in mathematics at some point enough, say >50 percent, would be verified, after which there would be interest among fellow mathematicians. But things went differently. Despite the fact that only a very small percentage of the Master's curriculum has been formalized, nowadays even Fields Medal winners use mechanically verified proofs,[8] because also for them the arguments used may be so complex that mechanical verification is needed. What can be used as the most appropriate logic is not yet clear. Mentioned facts place logic and foundations in a central position among the research and applications of mathematics.

The precision of the formal proofs explains in my view why mathematics is so very reliable. That informal proofs provide a very reasonable approximation is a particular feature of the human mind, being capable to provide them. I expect that a collaboration between human intuition and machine verification will bring mathematics and computer science to a new level of precision, without loosing their intellectual excitement. A major challenge is to make formalizing more user-friendly. Right now it is still very time-consuming. Already now occasionally a manuscript is submitted for publication that is

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6 Because many people have helped the formalization, especially a group of students in Vietnam, to some of which a PhD position was offered at the University of Pittsburgh.

7 The company Intel needed to reserve about 450M US$ for claims, because in 1994 they had brought a Pentium chip to the market which contained a bug. That gave a boost to machine authentication proving that a design meets a particular specification.

8 V. Voevodsky who initiated HoTT (2013), see also Voevodsky (2015). Furthermore T. Gowers, see Ganesalingam & Gowers (2013).

combined with a formalization of the proofs. In such cases a human referee will still be necessary, in order to determine whether the statement is formulated correctly and whether the results are interesting, for example because they can be related to other work. It can be safely expected that formally verified mathematics will bring the field and its practitioners to a next level of precision and beauty.

2. TWO KEYS TO MYSTICISM

Mysticism is often confused with mystification. This is because mystical experiences are difficult to express verbally. Mysticism therefore is often considered as irrational. Also, mystical experiences are deemed to be anti-rational, because rational thinking is held to be inadequate to induce mystical experiences; moreover, rational thinking is in fact correctly considered as a hindrance to mystical experiences. The Dutch philosopher Frits Staal (1975) has a more down to earth take on mystical experiences. He asserts that mysticism occurs in basically every culture. However, every culture has a different way of seeing and interpreting these mystical experiences. In addition, Staal holds that the mystical experience is neither rational, nor irrational: it refers to experience and as such it may be studied in a rational way.

2.1 Personal Domain

Induction

We set out to describe one of the possible personal keys to mystical experience: Buddhist insight meditation. However, first we need a few words on the phenomenon of 'consciousness'. It has two distinct features: 1. Consciousness is directed towards something, which is called the *object o*f consciousness. The object is based on data coming to us on the one hand through the physical senses, like for instance, through our ears, eyes, or on the other hand through the mind itself, like in the case of memories. 2. Then consciousness has a disposition, also known as the *mental state.* This determines the 'colouring' of consciousness. In fact the mental state, including its corresponding colour, determines the way the object is being processed and what direction the following actions will take. In the meditation practice a friendly discipline limits the stream of incoming objects as much as possible: in a quiet environment one sits still with eyes closed. In order to also quiet down the input coming from the mind, one may concentrate on the bodily movement caused by the neutral act of breathing. Nevertheless, sooner or later, one may get distracted and engage in a train of thought. Once this is noticed, attention is kindly directed back to the breathing. This practice, if systematically repeated (and if our lifestyle is adjusted accordingly), can give rise to a particular state of calm. This calm allows for a better control of our attention. In spite of this, through the emergence of particular mental states, several hindrances may occur, including desire, aversion, sleepiness and restlessness. Each of these hindrances is accompanied by a tendency to abandon the practice schedule. If one nevertheless continues to practice (observing 'Ah, there is pain', or 'Ah, there is restlessness'), a mental state may arise in which none of the hindrances is present. Such mental states indicate the beginning of certain mystical experiences.

The phenomena

The phenomena that typify a mystical experience are known to a large variety of traditions and are often described in terms of glory and rejoice. These traditions range from monotheistic religions such as Judaism, Christianity or Islam, poly- and a-theistic religions like Hinduism or Buddhism, and to shamanistic cultures. We here confine ourselves to the sometimes rather aloof descriptions of mystical states by classical Buddhism.

   The so called *jhānas (absorptions)* form part of what may be called the Buddhist mystic experience. During the first *jhāna* the state of consciousness of the practitioner contains the following substates: continuous attention, rapture, joy and concentration. Maintaining continuous attention requires an amount of energy and can be abandoned after a while, which will lead to the entering of the second *jhāna.* Rapture or ecstasy adversely give rise to some agitation. Dissociation of it may lead to the entering of the third *jhāna*. The third *jhāna* consists of merely joy and concentration. Once this joy is abandoned as well, sheer equanimity concentration remains. These different forms of *jhāna's* are consistent with 'mystical experiences' known from other cultures, such as the descriptions of Teresa of Avila versus that of Meister Eckhart.

   In any case, the mystical experience in all its varieties are soothing and beneficial for body and mind. It is also known that reducing stress may have many beneficial effects, including enhancing the functionality of the immune system. Subsequently, research on the effects of meditation has internationally seen an exponential growth.

2.2 Transpersonal

   There are indications that some of the phenomena that occur during a mystical state, can be explained by an increased concentration of neuromodulators in the brain, see e.g. Veening & Barendregt (2015). Especially β-endorphin has an overall calming and analgesic effect on mind and body. This is related to the fact that β-endorphin binds to the same neuroreceptors as morphine, which is the active component of opium. It seems that what can be experienced through personal endeavours in meditation, can be explained in terms of universal neurophysiological mechanisms of a group of molecules. The latter can thus be seen as transpersonal keys to mysticism.

2.3 Deepening

But wait a minute. Karl Marx (1844) gives a pause for thought here: 'Religion is the opium of the people'. According to Marx, religion---including mystical states---can be used by a ruling despot to delude and quiet down the masses. The above-mentioned explanation of the reduction of stress by endogenous opiates would even provide neurophysiological support for Marx's claim. However, there is more to say. In the tradition of insight meditation, the mystical experience is seen as a distraction as well: the goal is not temporary ecstasy and relief of uneasiness by suppression; rather the objective of the practice is to cultivate sustainable equanimity. The important Spanish mystic St. John of the Cross, whose path has been carefully compared to the path of the insight meditation by Meadow & Culligan (1987), writes that 'the way (...) does not consist of fun, experiences and spiritual feelings'. Temporary ecstatic experiences can become distractive addictions. They prevent the occurrence of wholesome sustainable changes in our personality.

Conditioning (personal)

What kind of changes are we to think of? All living organisms are 'conditioned'. Even monocellular ones: they avoid poison and approach nutrition. This is on a very primitive level the adaptive mechanism of desire and aversion. As another example, many insects are conditioned as follows: they navigate by maintaining a fixed angle to a bright light. When this source of light is at 'infinitely far', such as the sun or the moon, the insect will fly in a straight line and navigate efficiently. This adaptive conditioning, however, became life threatening from the moment that *homo sapiens* started to make fires. Light sources were no longer infinitely far. While maintaining a fixed angle towards these, insects will fly in a decreasing spiral towards the light source, eventually ending up inside the light source and burn.

Likewise, we human beings are conditioned, and not necessarily in an adaptive manner. We all know someone who tends to do things that he or she knows one shouldn't do, but can't resist doing them anyway. And being honest, we have to admit that we even sometimes do that ourselves. In other words, we are not free.

Deconditioning

Certain species of insects have evolved and learned not to fly into the flame. This adjustment happens on the time scale of evolution, over the course of many generations. By contrast, Homo sapiens, has a trump card using which one can decondition in the span of one lifetime. This happens in three phases.

1. Using strong concentration, that can be obtained by mentioned meditation exercises, we raise the resolution of observing our internal mental and bodily phenomena. At a certain point, we can experience that the stream of consciousness is not progressing in a continuous manner, but rather, that it is pulsating, like an impersonal mechanism. We do have intentions, but these are irrelevant, as also these are subjected to the patterns of the pulsating machine. Indeed, our intentions are not as independent as we might hold them to be. There is no fixed 'self' that governs this pattern. The experience of the absence of 'self' is called 'emptiness'. The self does exist, but merely as a construct made up of coordinated modules. It requires quite some energy to sustain the construct and coordination of the self. When the required energy

to sustain the construct is not available, like during periods of stress or illness, it may happen that the construct temporary collapses and emptiness is experienced.

2. One clings to the illusion of the fixed self in order to avoid the experience of emptiness. Nevertheless, once emptiness is experienced and perceived, its image is so vividly encrypted in us, that ignoring it is no longer an option. However, our inner resistance to the experience of emptiness is persistent and gives rise to suffering. After St. John of the Cross, this suffering is referred to as 'the dark night of the soul'. By means of systematic concentration and continuous mindfulness, one can learn to let go of the resistance to the experience of emptiness. At first this is temporarily. By letting go one experiences the so-called 'phase of equanimity.

3. Eventually we may see what lies at the root of our resistance and the consequent suffering, namely *wrong view*. It makes us to consider ourselves as operating in the centre of the universe using a fixed self in full control of things, notably ourselves. In order to let go of the resistance to the right view, one needs curiosity and mild surrender that consequently will cause our consciousness and behaviour to be more flexible.

It is almost a paradox that our conditioning decreases by acknowledging that we are fully conditioned! This can be explained as follows. Because we resist to the idea of being fully conditioned, we make all kinds of wrought manoeuvres driven by desire and aversion to the effect of covering up the state of not being in control. Abandoning the habit of compulsory generating this waste of energy, gives rise to calmness and new ways to organize our lives.

Towards a scientific explanation (transpersonal)

In cognitive neuroscience it is well known that all brain activity is determined by former neurophysiological activity and currently experienced input. Also, there is evidence that perception doesn't occur continuously, but in a pulsating manner, see Pascual-Marqui et al. (1995) and VanRullen et al. (2008). It is, however, less known, that these qualities can be experienced phenomenologically and that such experiences are difficult to accept.

This is because don't correspond with the view of self that we generally hold. The unmasking of the wrong view lies beneath the surface of any state of consciousness and is avoided by entertaining and maintaining a view of the self as a fixed and continuous entity. To this end, old self-affirming habits are employed frequently. A simple explanatory hypothesis is that the performance of these habits and tendencies yield a dose of endogenous opiates that keep the void (emptiness) out of sight. These habits are addictive and that is the reason why it can be so difficult to let go of old habits. Many tendencies and habits are characterised by 'replacing fear for no-thing, by fear for some-thing', as the psychologist Rollo May (1950) neatly phrases it. This has as ultimate consequence that homo sapiens may cherish irrational anxieties and thoughts and even initiate wars and severely pollute the environment. The above-mentioned addiction also explains why at this precarious moment in the history of mankind, politicians persistently remain incapable of taking the necessary measures, that obviously will serve the benefit of all.

We will now leave the cause of suffering behind and proceed with the mentioned solution. Is this solution plausible? Several authors, including Zylberberg et al. (2011), propose the brain as a hybrid Turing machine: pulsating according to prefixed patterns, while a neural net determines the intermediate steps. This view of the functioning of the brain is consistent with the meditative experience of observing consciousness as a

deterministic pulsating stream of input. Barendregt & Raffone (2013) further extend this proposed model to the possibility of observing one's own mental state of consciousness. The latter is essential to mindfulness. With the development of mindfulness, an extra sense is cultivated that acts as a radar for our automatic pilot. The new developed sense allows for the possibility of adjusting the internalised mental programmes. This explanation suffices for the possibility of letting go of persisting habits, including the clinging to the wrong view of self. The currently flourishing field of meditation research will hopefully before too long be able to establish results in the said direction. Intuition based on meditation experience may act in a similar way for neuroscience as intuition acts in mathematics: as a source for formulating hypotheses that can consequently be followed up and verified experimentally. However, before that may happen, at present humanity finds itself collectively in a state of 'dark night'. Therefore there is urgency. But there is confidence that there will be light at the end of the tunnel.

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