

# Lambda Calculus

We introduce terms to represent arithmetic in  $\lambda$ -calculus

NOTATION For terms  $F, A \in \Lambda$  and  $n \in \mathbb{N}$  define  $F^n A$  as follows

$$F^0 A \triangleq A$$

$$F^{k+1} A \triangleq F(F^k A)$$

Thus  $F^2 A = F(F A)$ ,  $F^3 A = F(F(F A))$

DEFINITION (i) The *Church numerals* are  $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \dots$  with

$$\mathbf{c}_n \triangleq \lambda f x. f^n x$$

(ii) A function  $f: \mathbb{N}^k \rightarrow \mathbb{N}$  is called  *$\lambda$ -definable* if there is a term  $F \in \Lambda$  such that for all  $\vec{n} \in \mathbb{N}^k$  one has

$$F \mathbf{c}_{n_1} \dots \mathbf{c}_{n_k} = \mathbf{c}_{f(n_1, \dots, n_k)}$$

## Some representable functions

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Define

$$F_+ = \lambda n m \lambda f x . n f (m f x)$$

$$F_* = \lambda n m \lambda f x . n (m f) x$$

$$F_{\text{exp}} = \lambda n m \lambda f x . m n f x$$

These functions  $\lambda$ -define addition, multiplication, and exponentiation

$$F_+ \mathbf{c}_n \mathbf{c}_m =_{\beta} \mathbf{c}_{n+m}$$

$$F_* \mathbf{c}_n \mathbf{c}_m =_{\beta} \mathbf{c}_{n \cdot m}$$

$$F_{\text{exp}} \mathbf{c}_n \mathbf{c}_m =_{\beta} \mathbf{c}_{m^n}$$

We verify this only for  $F_+$

$$F_+ \mathbf{c}_n \mathbf{c}_m =_{\beta} \lambda f x . \mathbf{c}_n f (\mathbf{c}_m f x) =_{\beta} \lambda f x . f^n (f^m x) \equiv \lambda f x . f^{n+m} x \equiv \mathbf{c}_{n+m}$$

Corollary. The function  $f: \mathbb{N} \rightarrow \mathbb{N}$ , with  $f(n) = (n + 2)^3$ , is  $\lambda$ -definable

## The predecessor function

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This is  $p: \mathbb{N} \rightarrow \mathbb{N}$  with  $p(0) = 0$ ,  $p(n + 1) = n$

Define

$$\begin{aligned}\langle M, N \rangle &\triangleq \lambda z. zMN \\ \text{true} &\triangleq \mathbf{K} \triangleq \lambda x_1 x_2. x_1 \\ \text{false} &\triangleq \mathbf{K}_* \triangleq \lambda x_1 x_2. x_2 \\ \mathbf{P}_1 &\triangleq \lambda a. a\mathbf{K} \\ \mathbf{P}_2 &\triangleq \lambda a. a\mathbf{K}_*\end{aligned}$$

Then we can find the components from a pair:

$$\mathbf{P}_i \langle M_1, M_2 \rangle = \langle M_1, M_2 \rangle (\lambda x_1 x_2. x_i) = (\lambda x_1 x_2. x_i) M_1 M_2 = M_i$$

## Construction

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In order to  $\lambda$ -define the predecessor  $p$  we do it for “ $n \rightsquigarrow \langle n, p(n) \rangle$ ”

i.e.  $0 \rightsquigarrow \langle 0, 0 \rangle$ ,  $1 \rightsquigarrow \langle 1, 0 \rangle$ ,  $2 \rightsquigarrow \langle 2, 1 \rangle$ ,  $3 \rightsquigarrow \langle 3, 2 \rangle \dots$

Let  $T = \lambda p. \langle F_+ (P_1 p) \mathbf{c}_1, P_1 p \rangle$ . Then  $T \langle \mathbf{c}_n, \mathbf{c}_m \rangle = \langle \mathbf{c}_{n+1}, \mathbf{c}_n \rangle$

Note that

$$\langle \mathbf{c}_0, \mathbf{c}_0 \rangle \rightsquigarrow^T \langle \mathbf{c}_1, \mathbf{c}_0 \rangle \rightsquigarrow^T \langle \mathbf{c}_2, \mathbf{c}_1 \rangle \rightsquigarrow^T \langle \mathbf{c}_3, \mathbf{c}_2 \rangle \rightsquigarrow^T \dots$$

Hence  $T^k \langle \mathbf{c}_0, \mathbf{c}_0 \rangle = \langle \mathbf{c}_k, \mathbf{c}_{k-1} \rangle$  Taking  $P \triangleq \lambda n. P_2(nT \langle \mathbf{c}_0, \mathbf{c}_0 \rangle)$  one has

$$\begin{aligned} P\mathbf{c}_0 &\equiv (\lambda n. P_2(nT \langle \mathbf{c}_0, \mathbf{c}_0 \rangle))\mathbf{c}_0 \\ &= P_2(\mathbf{c}_0 T \langle \mathbf{c}_0, \mathbf{c}_0 \rangle) \\ &= P_2 \langle \mathbf{c}_0, \mathbf{c}_0 \rangle \\ &= \mathbf{c}_0 \equiv \mathbf{c}_{p(0)} \\ P\mathbf{c}_{k+1} &\equiv (\lambda n. P_2(nT \langle \mathbf{c}_0, \mathbf{c}_0 \rangle))\mathbf{c}_{k+1} \\ &= P_2(T^{k+1} \langle \mathbf{c}_0, \mathbf{c}_0 \rangle) \\ &= P_2 \langle \mathbf{c}_{k+1}, \mathbf{c}_k \rangle \\ &= \mathbf{c}_k \equiv \mathbf{c}_{p(k+1)} \end{aligned}$$

Therefore  $P\mathbf{c}_k = \mathbf{c}_{p(k)}$  for all  $k \in \mathbb{N}$ . Hence  $p$  is  $\lambda$ -definable by  $P$

## Church's Thesis

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Thesis: Let the function  $f: \mathbb{N}^k \rightarrow \mathbb{N}$  be given.

Then  $f$  is computable by a human if and only if it is  $\lambda$ -definable

Then Church went on constructing a function that is not  $\lambda$ -definable  
hence by his thesis, not human computable

## Data types

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We give two examples: natural numbers and trees

Natural numbers:

$$\text{Nat} := \text{zero} \mid \text{suc Nat}$$
$$\text{Tree} := \text{leaf} \mid \text{pair Tree Tree}$$

Equivalently, as a context-free grammar

$$\text{Nat} \rightarrow z \mid (s \text{ Nat})$$
$$\text{Tree} \rightarrow l \mid (p \text{ Tree Tree})$$

We know what belongs to it

$$\text{Nat} = \{z, (sz), (s(sz)), (s(s(sz))), \dots\} = \{s^n z \mid n \in \mathbb{N}\}$$

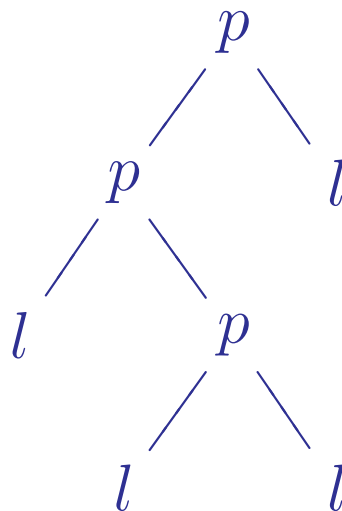
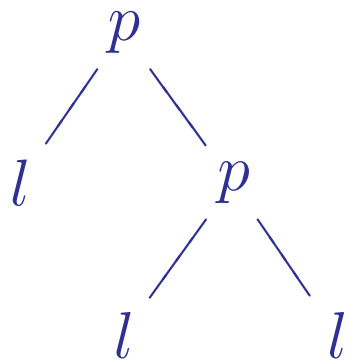
# Trees

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Tree :=  $l \mid (p \text{ Tree Tree})$

Examples of elements of (language defined by) Tree

$(pl(p11))$     and     $(p(p1(p11))1)$





# Translating data into lambda terms (Böhm-Berarducci)

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Nat:  $t \rightsquigarrow \lceil t \rceil := \lambda sz.t$

For example

$$\lceil (s(s(sz))) \rceil = \lambda sz.(s(s(sz))) = \mathbf{c}_3$$

Tree:  $t \rightsquigarrow \lceil t \rceil := \lambda pl.t$

For example

$$\lceil (pl(pl1)) \rceil = \lambda pl.(pl(pl1))$$

## Operating on data after representing them

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For Nat we could operate on the codes to represent wanted functions:

$$F_+ \ulcorner n \urcorner \ulcorner m \urcorner =_{\beta} \ulcorner n + m \urcorner$$
$$F_{\times} \ulcorner n \urcorner \ulcorner m \urcorner =_{\beta} \ulcorner n \times m \urcorner$$

Define on Trees the operation of mirroring:

$$\text{Mirror } (1) = 1$$
$$\text{Mirror } (p \ t1 \ t2) = (p \ (\text{Mirror } t2) \ (\text{Mirror } t1))$$

We will construct a  $\lambda$ -term  $F_M$  such that

$$F_M \ulcorner t \urcorner = \ulcorner \text{Mirror}(t) \urcorner$$

## Representing the basic operation on Tree

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LEMMA. There exists a  $P \in \Lambda$  such that

$$P \ulcorner t_1 \urcorner \ulcorner t_2 \urcorner = \ulcorner p t_1 t_2 \urcorner \quad (1)$$

PROOF. Taking  $P := \lambda t_1 t_2 p1. p(t_1 p1)(t_2 p1)$  we claim that (1) holds.

Note that  $t \in \text{Tree}$  can be considered as a  $\lambda$ -term:  $\text{Tree} \subseteq \Lambda$

Since  $\ulcorner t \urcorner = \lambda p1. t$  one has  $\ulcorner t \urcorner p1 =_{\beta} t$ . Hence

$$\begin{aligned} P \ulcorner t_1 \urcorner \ulcorner t_2 \urcorner &= (\lambda t_1 t_2 p1. p(t_1 p1)(t_2 p1)) \ulcorner t_1 \urcorner \ulcorner t_2 \urcorner \\ &= \lambda p1. p(\ulcorner t_1 \urcorner p1)(\ulcorner t_2 \urcorner p1) \\ &= \lambda p1. p t_1 t_2 \\ &= \ulcorner p t_1 t_2 \urcorner. \blacksquare \end{aligned}$$

PROPOSITION. There exists an  $F_M \in \Lambda$  such that for all  $t \in \text{Tree}$

$$F_M \ulcorner t \urcorner =_{\beta} \ulcorner \text{Mirror}(t) \urcorner \quad (2)$$

PROOF. Take  $F_M \triangleq \lambda t p l. t p' l$ , where  $p' \triangleq \lambda a b. p b a$ .

We claim by induction that (2) holds. **Note that  $F_M \ulcorner t \urcorner p l = \ulcorner t \urcorner p' l$ .**

Case  $t=1$ . Then

$$F_M \ulcorner 1 \urcorner = \lambda p l. (\lambda p l. 1) p' l = \lambda p l. 1 = \ulcorner 1 \urcorner = \text{Mirror}(\ulcorner 1 \urcorner).$$

Case  $t = p t_1 t_2$ . Then

$$\begin{aligned} F_M \ulcorner p t_1 t_2 \urcorner &= \lambda p l. \ulcorner p t_1 t_2 \urcorner p' l \\ &= \lambda p l. P \ulcorner t_1 \urcorner \ulcorner t_2 \urcorner p' l \\ &= \lambda p l. p' (\ulcorner t_1 \urcorner p' l) (\ulcorner t_2 \urcorner p' l) \\ &= \lambda p l. p (\ulcorner t_2 \urcorner p' l) (\ulcorner t_1 \urcorner p' l) \\ &= \lambda p l. p (F_M \ulcorner t_2 \urcorner p l) (F_M \ulcorner t_1 \urcorner p l) \\ &= \lambda p l. p (\ulcorner \text{Mirror}(t_2) \urcorner p l) (\ulcorner \text{Mirror}(t_1) \urcorner p l), \quad \text{by the IH,} \\ &= \ulcorner p (\text{Mirror}(t_2)) (\text{Mirror}(t_1)) \urcorner \\ &= \ulcorner \text{Mirror}(p t_1 t_2) \urcorner. \blacksquare \end{aligned}$$