Lambda Calculus final

Form of λ-terms

These are

$$x \mid PQ \mid \lambda x.P$$

(variable, application, abstraction)

Therefore every term M is of the form

$$M \equiv \lambda x_1 \ldots x_n.P$$

where P is a variable or an application

CLAIM. Let P be not an abstraction

Then P is of one of the following forms, with $0 \le m$

$$\begin{array}{rcl} P & \equiv & yQ_1 \ldots Q_m \\ P & \equiv & (\lambda y.R_0)R_1Q_1 \ldots Q_m \end{array}$$

Therefore every term M is of the form

$M \equiv \lambda x_1 \ldots x_n.yQ_1 \ldots Q_m$ y is the *head variable*

$M \equiv \lambda x_1 \ldots x_n.(\lambda y.R_0)R_1Q_1 \ldots Q_m$ $(\lambda y.R_0)R_1$ is the *head redex*

Being and having a head normal form

Let M be a λ-term. Then M *is* a head normal form if

$$M \equiv \lambda x_1 \ldots x_n . y Q_1 \ldots Q_m$$

M *has* a head normal form if $M =_\beta M'$ and M' is a head normal form

$\mathsf{I}, \mathsf{K}, \mathsf{S}$ are head normal forms

Y is not a hnf

Y has a hnf

Ω has no hnf

Non-definability of coding

We have constructed a term H (werkcollege 08.01.2014) such that

$$\begin{aligned} H\ulcorner M\urcorner &= \text{true}, \qquad \text{if } M \text{ is a hnf}, \\ &= \text{false}, \qquad \text{if } M \text{ is not a hnf}. \end{aligned}$$

Hence

$$\begin{aligned} H\ulcorner \lambda x.\mathsf{I}x\urcorner &= \text{true}, \\ H\ulcorner \lambda x.x\urcorner &= \text{false}. \end{aligned}$$

But note that $\lambda x.\mathsf{I}x =_\beta \lambda x.x$.
CONSEQUENCE. There is no term C (for coding) such that

$$CM = \ulcorner M\urcorner.$$

Proof. Suppose C exists. Then

$$\begin{aligned} \text{true} &= H\ulcorner \lambda x.\mathsf{I}x\urcorner \\ &= H(C\lambda x.\mathsf{I}x) = H(C\lambda x.x) \\ &= H\ulcorner \lambda x.x\urcorner = \text{false}, \end{aligned}$$

a contradiction.