Uitwerking Lambda Calculus (week 6, 08.01.2014)

1. (a) Given λ-terms P, Q, construct a λ-term $F_{P,Q}$ such that

$$\begin{aligned} F_{P,Q}\mathbf{true} &= P; \\ F_{P,Q}\mathbf{false} &= Q. \end{aligned}$$

Oplossing. Zij $F_{P,Q} = \lambda x.xPQ = \langle P, Q\rangle$.
Dan $F_{P,Q}\mathbf{true} = \mathbf{true}\ PQ = P$ en $F_{P,Q}\mathbf{false} = \mathbf{false}\ PQ = Q$

(b) Construct a λ-term F_{neg} such that

$$F_{\text{neg}}\mathbf{true} = \mathbf{false}\ \&\ F_{\text{neg}}\mathbf{false} = \mathbf{true}.$$

Dit is een speciaal geval van (a) met $P = \mathbf{false}$ en $Q = \mathbf{true}$. Dus $F_{neg} = \langle \mathbf{false}, \mathbf{true}\rangle$.

(c) Construct a term F_{and} such that

$$\begin{aligned} F_{\text{and}}\mathbf{true\ true} &= \mathbf{true}; \\ F_{\text{and}}\mathbf{true\ false} &= \mathbf{false}; \\ F_{\text{and}}\mathbf{false\ true} &= \mathbf{false}; \\ F_{\text{and}}\mathbf{false\ false} &= \mathbf{false}. \end{aligned}$$

Neem $F_{and} = \lambda xy.xyx$. Dan $F_{and}\ \mathbf{true} = \lambda y.\mathbf{true}\ y\mathbf{true} = \lambda y.y = \mathbf{I}$ en $F_{and}\ \mathbf{false} = \lambda y.\mathbf{false}\ y\ \mathbf{false} = \lambda y.\mathbf{false}$.
Dus F_{and} werkt zoals de bedoeling is.
Een andere mogelijkheid is $F_{and} = \langle\langle \mathbf{true}, \mathbf{false}\rangle, \langle \mathbf{false}, \mathbf{false}\rangle\rangle$.

2. (a) Construct a λ-term G such that

$$\begin{aligned} G\ulcorner x\urcorner &= \mathbf{true} \\ G\ulcorner PQ\urcorner &= \mathbf{false} \\ G\ulcorner \lambda x.P\urcorner &= \mathbf{false}. \end{aligned}$$

Oplossing. Probeer $G = \langle\langle B_1, B_2, B_3\rangle\rangle$ voor zekere B_1, B_2, B_3. Dan
$G\ulcorner x\urcorner = \ulcorner x\urcorner\langle B_1, B_2, B_3\rangle = \langle B_1, B_2, B_3\rangle U_1^3 x\langle B_1, B_2, B_3\rangle = B_1 x\langle B_1, B_2, B_3\rangle$, dus neem $B_1 = \lambda xy.\mathbf{true}$
$G\ulcorner PQ\urcorner = \ulcorner PQ\urcorner\langle B_1, B_2, B_3\rangle = \langle B_1, B_2, B_3\rangle U_2^3\ulcorner P\urcorner\ulcorner Q\urcorner\langle B_1, B_2, B_3\rangle = B_2\ulcorner P\urcorner\ulcorner Q\urcorner\langle B_1, B_2, B_3\rangle$. Neem dus $B_2 = \lambda xyz.\mathbf{false}$.
$G\ulcorner \lambda x.P\urcorner = \langle B_1, B_2, B_3\rangle U_3^3(\lambda x.\ulcorner P\urcorner)\langle B_1, B_2, B_3\rangle = B_3(\lambda x.\ulcorner P\urcorner)\langle B_1, B_2, B_3\rangle$. Dus $B_3 = \lambda xy.\mathbf{false}$.
Conclusie: $G = \langle\langle \lambda xy.\mathbf{true}, \lambda xyz.\mathbf{false}, \lambda xy.\mathbf{false}\rangle\rangle$

(b) Construct a λ-term V such that

$$\begin{aligned} V\ulcorner x \urcorner &= \mathbf{true} \\ V\ulcorner PQ \urcorner &= V\ulcorner P \urcorner \\ V\ulcorner \lambda x.P \urcorner &= \mathbf{false}. \end{aligned}$$

Oplossing. Omdat V en G overeenkomen op $\ulcorner x \urcorner$ en $\ulcorner \lambda x.M \urcorner$, geldt dat hun B_1 en B_3 ook gelijk zijn.

$V\ulcorner PQ \urcorner = B_2\ulcorner P \urcorner\ulcorner Q \urcorner\langle B_1, B_2, B_3\rangle = V\ulcorner P \urcorner = \langle\langle B_1, B_2, B_3\rangle\rangle\ulcorner P \urcorner = \ulcorner P \urcorner\langle B_1, B_2, B_3\rangle$. Dus $B_2 = \lambda xyz.xz$.

Conclusie: $V = \langle\langle \lambda xy.\mathbf{true}, \lambda xyz.xz, \lambda xy.\mathbf{false}\rangle\rangle$.

(c) Construct a λ-term H such that

$$\begin{aligned} H\ulcorner x \urcorner &= \mathbf{true} \\ H\ulcorner PQ \urcorner &= V\ulcorner P \urcorner \\ H\ulcorner \lambda x.P \urcorner &= H\ulcorner P \urcorner. \end{aligned}$$

Oplossing. Weer: $B_1 = \lambda xy.\mathbf{true}$. $H\ulcorner PQ \urcorner = B_2\ulcorner P \urcorner\ulcorner Q \urcorner\langle B_1, B_2, B_3\rangle = V\ulcorner P \urcorner$. Dus $B_2 = \lambda xyz.Vx$

$H\ulcorner \lambda x.P \urcorner = B_3(\lambda x.\ulcorner P \urcorner)\langle B_1, B_2, B_3\rangle = H\ulcorner P \urcorner$. Dus $B_3 = \lambda yz.yxz$.

Conclusie: $H = \langle\langle \lambda xy.\mathbf{true}, \lambda xyz.Vx, \lambda yz.yxz\rangle\rangle$.

(d) Compute $H\ulcorner S \urcorner$, $H\ulcorner \mathsf{Y} \urcorner$, where $\mathsf{Y} \triangleq \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$.
Oplossing. $\mathsf{S} \triangleq \lambda abc.ac(bc)$, dus

$$\begin{aligned} H\ulcorner \mathsf{S} \urcorner &= H\ulcorner \lambda abc.ac(bc) \urcorner \\ &= H\ulcorner \lambda bc.ac(bc) \urcorner \\ &= H\ulcorner \lambda c.ac(bc) \urcorner \\ &= H\ulcorner ac(bc) \urcorner \\ &= V\ulcorner ac \urcorner \\ &= V\ulcorner a \urcorner \\ &= \mathbf{true}. \end{aligned}$$

$H\ulcorner \mathsf{Y} \urcorner = H\ulcorner \lambda f.PQ \urcorner = H\ulcorner PQ \urcorner = V\ulcorner \lambda x.f(xx) \urcorner = \mathbf{false}$.

(e) Define $\mathsf{Y}' \triangleq \lambda f.f((\lambda x.f(xx))(\lambda x.f(xx))$. Note that $\mathsf{Y} = \mathsf{Y}'$. Compute $H\ulcorner \mathsf{Y}' \urcorner$. Oplossing.

$$\begin{aligned} H\ulcorner \mathsf{Y}' \urcorner &= H\ulcorner \lambda f.f(PQ) \urcorner \\ &= H\ulcorner f(PQ) \urcorner \\ &= V\ulcorner f \urcorner \\ &= \mathbf{true}. \end{aligned}$$

3. (a) Show that Every λ-term M is either of the form

$$M \equiv \lambda x_1 \dots x_n . y Q_1 \dots Q_m, \ 0 \leq n, m \tag{1}$$

(with *head variable* y) or

$$M \equiv \lambda x_1 \dots x_n . (\lambda y.P) Q_0 Q_1 \dots Q_m, \ 0 \leq n, m \tag{2}$$

(with *head redex* $(\lambda y.P)Q$). We bewijzen dit met inductie.

- Geval 1. $M = x$ dan is M van de vorm (1) met $n = m = 0$.
- Geval 2. $M \equiv NL$. Stel N, L zijn van de vorm (1) of (2). Dan $N \equiv \lambda x_1 \dots x_n . \Box Q_1 \dots Q_m$, met $\Box$ van de vorm y of $(\lambda y.P)Q_0$.
  Geval 2.1. $n = 0$. Dan $NL = \Box Q_1 \dots Q_m L$ is van de vorm (1) of (2) (geen initiële λ-s).
  Geval 2.2. $n > 0$. Dan $NL \equiv (\lambda x_1 \dots x_n . \Box Q_1 \dots Q_m)L$ is van de vorm (2) (weer geen initiële λ-s).
- Geval 3. $M \equiv \lambda x.N$. Stel N is van de vorm (1) of (2)

  $$N \equiv \lambda x_1 \dots x_n . \Box Q_1 \dots Q_m.$$

  Dan is $M = \lambda x x_1 \dots x_n . \Box Q_1 \dots Q_m$ weer van die vorm.
  Hiermee is de uitspraak bewezen.

(b) Show that

$$\begin{array}{lcll} H \ulcorner M \urcorner & = & \mathbf{true}, & \text{if } M \text{ is of form (1);} \\ H \ulcorner M \urcorner & = & \mathbf{false}, & \text{if } M \text{ is of form (2).} \end{array}$$

Oplossing. Schrijf

$$M \equiv \lambda x_1 \dots x_n . \Box Q_1 \dots Q_m.$$

H eet alle λx die vooraan staan op, dus
$H \ulcorner \lambda x_1 \dots x_n . \Box Q_1 \dots Q_m \urcorner = H \ulcorner \Box Q_1 \dots Q_m \urcorner = V \ulcorner \Box \urcorner$. Verder is

$$\begin{array}{lcll} V \ulcorner \Box \urcorner & = & V \ulcorner y \urcorner = \mathbf{true}, & \text{als } \Box \equiv y; \\ V \ulcorner \Box \urcorner & = & V \ulcorner \lambda y.P \urcorner = \mathbf{false}, & \text{als } \Box \equiv (\lambda y.P)\vec{Q}. \end{array}$$