Degrees of undecidability of in Term Rewriting

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Overview

- Term Rewriting Systems (TRS): definitions and properties
- Overview of results
- The arithmetic and analytical hierarchy
- Technical details
  - Relating Turing Machines to TRSs
  - Classification of properties of Turing Machines
  - Weak Church-Rosser
  - Church-Rosser
  - Dependency Pair Problems

Term Rewriting Systems (TRS): definitions and properties

- A Signature Σ is a finite set of symbols f each having a fixed arity.
- The set $Ter(\Sigma, \mathcal{X})$ of terms is the smallest set satisfying:
  - $\mathcal{X} \subseteq Ter(\Sigma, \mathcal{X})$, and
  - $f(t_1, \ldots, t_n) \in Ter(\Sigma, \mathcal{X})$ if $f \in \Sigma$ with arity n and $\forall i : t_i \in Ter(\Sigma, \mathcal{X})$.
- A term rewriting system (TRS) over Σ, $\mathcal{X}$ is a finite set R of pairs $\langle \ell, r \rangle \in Ter(\Sigma, \mathcal{X})$, called rewrite rules usually written as $\ell \rightarrow r$ for which
  - the left-hand side ℓ is not a variable ($\ell \notin \mathcal{X}$)
  - all variables in the right-hand side r occur in ℓ ($Var(r) \subseteq Var(\ell)$).

Term Rewriting Systems (TRS): definitions and properties

For terms $s, t \in Ter(\Sigma, \mathcal{X})$ we write $s \to_R t$ if there exists a rule $\ell \to r \in R$, a substitution σ and a context ('term with a hole') C such that $s \equiv C[\ell\sigma]$ and $t \equiv C[r\sigma]$

- $\to_R$ is the rewrite relation induced by R,
- $\leftrightarrow_R$ denotes the symmetric, reflexive closure of $\to_R$.
- $\to_R^+$ denotes the transitive closure of $\to_R$.
- $\to_R^*$ denotes the reflexive, transitive closure of $\to_R$.

Basic TRS properties

- R is strongly normalizing (or terminating) on t, denoted $\mathsf{SN}_R(t)$,
  if every rewrite sequence starting from t is finite.
- R is confluent (or Church-Rosser) on t, denoted $\mathsf{CR}_R(t)$,
  if every pair of finite coinitial reductions starting from t can be extended to a common reduct, that is,
  $\forall t_1, t_2.\ t_1 \leftarrow^* t \rightarrow^* t_2 \Rightarrow \exists d.\ t_1 \rightarrow^* d \leftarrow^* t_2$.
- R is weakly confluent (or weakly Church-Rosser) on t, denoted $\mathsf{WCR}_R(t)$, if every pair of coinitial rewrite steps starting from t can be joined, that is,
  $\forall t_1, t_2.\ t_1 \leftarrow t \longrightarrow t_2 \Rightarrow \exists d.\ t_1 \rightarrow^* d \leftarrow^* t_2$.

R is strongly normalizing (SN_R), confluent (CR_R) or weakly confluent (WCR_R) if the respective property holds on all terms $t \in \mathit{Ter}(\Sigma, \mathcal{X})$.

TRS properties

Church-Rosser and Weak Church-Rosser are usually also considered on the ground terms only (ground = closed; no free variables).

- R is ground Church-Rosser, denoted grCR_R, if every pair of finite coinitial reductions starting from any ground t can be extended to a common reduct, that is, $\forall t, t_1, t_2$ ground. $t_1 \leftarrow^* t \rightarrow^* t_2 \Rightarrow \exists d.\, t_1 \rightarrow^* d \leftarrow^* t_2$.
- R is ground weakly Church-Rosser, denoted grWCR_R, if every pair of coinitial rewrite steps starting from a ground t can be joined, that is, $\forall t, t_1, t_2$ ground. $t_1 \leftarrow t \longrightarrow t_2 \Rightarrow \exists d.\, t_1 \rightarrow^* d \leftarrow^* t_2$.

Undecidability of TRS properties

All interesting properties about TRSs are undecidable, but how undecidable?

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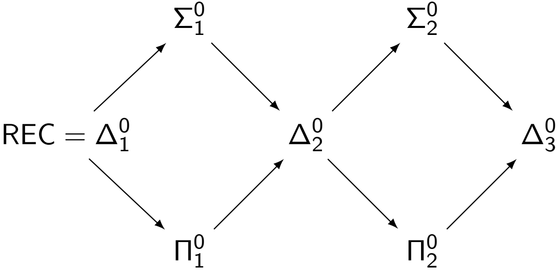
	SN	WN	CR	grCR	WCR	grWCR	DP	DP^{min}
uniform	Π^0_2	Π^0_2	Π^0_2	Π^0_2	Σ^0_1	Π^0_2	Π^1_1	Π^0_2
single term	Σ^0_1	Σ^0_1	Π^0_2	Π^0_2	Σ^0_1	Σ^0_1	Π^1_1	–

Existing work: Huet and Lankford (1978)
Independent (but published earlier): J.G Simonsen (2009)
New Contributions in red

The Arithmetic Hierarchy



REC = class of decidable problems (over the natural numbers),
$\Sigma_1^0 := \exists\text{REC}$, $\Pi_1^0 := \forall\text{REC}$, $\Sigma_2^0 := \exists\forall\text{REC}$, $\Pi_2^0 := \forall\exists\text{REC}$, etc.

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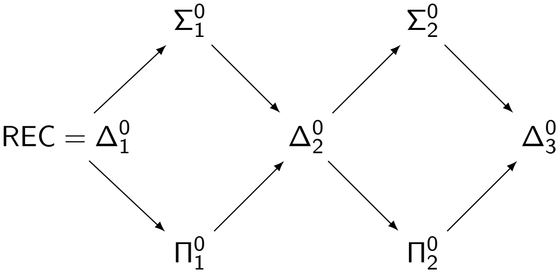

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$\Delta_n^0 := \Sigma_n^0 \bigcap \Pi_n^0$.
$\Sigma_n^0 = \{A \mid \overline{A} \in \Pi_n^0\}$, $\Pi_n^0 = \{A \mid \overline{A} \in \Sigma_n^0\}$

Examples

We leave encodings implicit, so we say e.g.

- $t \to^* q := \exists \langle s_1, \ldots, s_n \rangle (t \longrightarrow_R s_1 \longrightarrow_R \ldots \longrightarrow_R s_n = q)$ is in Σ^1_0.
- $T(M, \langle \vec{x} \rangle, u, v) :=$ m is a Turing Machine M, u is the computation of M on $\vec{x}$ whose end result is v
  is in REC. Kleene's T-predicate.
- $\mathrm{TOTAL}(M) := \forall x \exists u, v T(m, \langle x \rangle, u, v)$ is in Π^0_2.

Properties of the classes in the Arithmetic Hierarchy

Any formula is equivalent to a formula in prenex normal form

- $Qx\,(\varphi) \otimes Qy\,(\psi) \iff Qx\,Q\,y\,(\varphi \otimes \psi)$, for $\otimes \in \{\wedge, \vee\}$, $Q \in \{\forall, \exists\}$.
- $Qx\,(\varphi) \rightarrow Qy\,(\psi) \iff \overline{Q}x\,Qy\,(\varphi \rightarrow \psi)$, for $Q \in \{\forall, \exists\}$.
  $\iff Qy\,\overline{Q}x\,(\varphi \rightarrow \psi)$.

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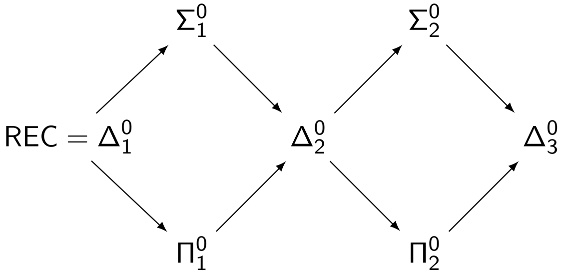
Compression of quantifiers of the same type. Symbolically:

- $\forall\forall \mapsto \forall$ and $\exists\exists \mapsto \exists$
  $\forall x \forall y (P(x, y)) \iff \forall z (P((z)_1, (z)_2))$

A bounded quantifier is no quantifier:

- $\forall x < n\,\mathrm{REC} = \mathrm{REC}$,
- $\exists x < n\,\mathrm{REC} = \mathrm{REC}$

The Arithmetic Hierarchy



Theorem $\Sigma_i^0 \subsetneq \Delta_{i+1}^0 \subsetneq \Sigma_{n+1}^0$ and $\Pi_i^0 \subsetneq \Delta_{i+1}^0 \subsetneq \Pi_{n+1}^0$

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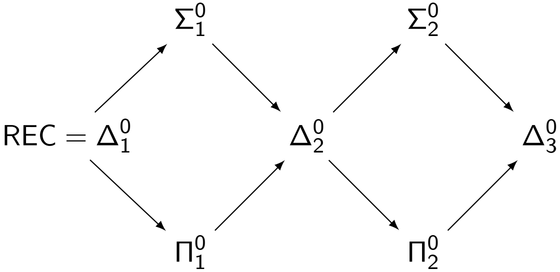

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$\text{BlankTape}(M) := \exists u, v\ T(M, \langle\rangle, u, v) \in \Sigma_1^0 \setminus \Pi_1^0$

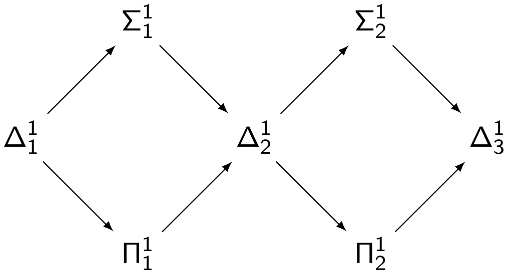
$\text{TOTAL}(M) := \forall x \exists u, v\ T(M, \langle x \rangle, u, v) \in \Pi_2^0 \setminus \Sigma_2^0$

Above the arithmetical hierarchy: analytical hierarchy

All properties definable in first order arithmetic reside in the arithmetical hierarchy.
If we want to quantify over functions from $\mathbb{N}$ to $\mathbb{N}$ (infinite sequences of numbers), we end up in the analytical hierarchy.
Function variables are usually α, β, etc.
Example:

$$\exists\alpha\forall i(\alpha(i) \rightarrow_R \alpha(i+1))$$

The Analytic Hierarchy



$\Sigma^1_1 := \exists\alpha\forall x\text{REC}$, $\Pi^1_1 := \forall\alpha\exists x\text{REC}$, $\Sigma^1_2 := \exists\beta\forall\alpha\exists x\text{REC}$, etc.
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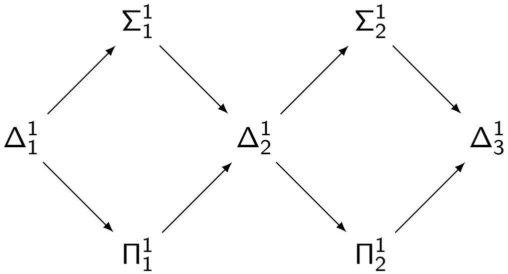

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$\Delta^1_n := \Sigma^1_n \bigcap \Pi^1_n$.
$\Sigma^1_{n+1} = \exists^1\alpha\Pi^1_n$, $\Pi^1_{n+1} = \exists\beta\Sigma^1_n$
$\Sigma^1_n = \{A \mid \overline{A} \in \Pi^1_n\}$, $\Pi^1_n = \{A \mid \overline{A} \in \Sigma^1_n\}$.
Theorem $\Sigma^1_i \subsetneq \Delta^1_{i+1} \subsetneq \Sigma^1_{n+1}$ and $\Pi^1_i \subsetneq \Delta^1_{i+1} \subsetneq \Pi^1_{n+1}$
$\mathrm{WF}(M) :=$ "M defines a well-founded relation $>_M$" $\in \Pi^1_1 \setminus \Sigma^1_1$

Properties of the classes in the Analytic Hierarchy

We have quantifiers over numbers $\forall, \exists$ and over functions $\forall^1, \exists^1$. A number of quantifiers of the same type can be compressed into one.

- $\forall^1\forall^1 \mapsto \forall^1$ and $\exists^1\exists^1 \mapsto \exists^1$

$\forall^1$ subsumes $\forall$.

- $\forall^1\forall \mapsto \forall^1$ and $\exists^1\exists \mapsto \exists^1$

$\forall^1$ moves outside over $\exists$ and $\exists^1$ moves outside over $\forall$.

- $\exists\forall^1 \mapsto \forall^1\exists$ and $\forall\exists^1 \mapsto \exists^1\forall$
- The standard form of an element of the analytic hierarchy is $Q_1^1 Q_2^1 \ldots Q_n^1 Q$ with swopping quantifiers and Q opposite to Q_n^1.

Proving that a property is essentially Π^0_2 (and not "lower")

A total recursive function f many-one reduces problem A to problem B if

$$A(x) \Longleftrightarrow B(f(x)), \text{ for all } x$$

So "if we want to decide $A(x)$, we only have to decide $B(x)$".

$$A \ll_m B \quad (A \text{ is many-one reducible to } B)$$

in case such an f exists.

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Definition
B is called Π^0_2-complete if $B \in \Pi^0_2$ and forall $A \in \Pi^0_2$, $A \ll_m B$.
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Theorem
BlankTape(M) is Σ^0_1-complete,
TOTAL(M) is Π^0_2-complete,
WF(M) is Π^1_1-complete.

To prove that WCR is Σ^0_1-complete:
Reduce it to BlankTape

From Turing Machines to TRSs

Translating a Turing machine $M = (Q, \Sigma, q_0, \delta)$ to a TRS R_M
Function symbols:

$a \in \Sigma$	$\mapsto$	unary function $a(-)$
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Configurations:
Right of the reading head: $a\,b\,a\,a\,\Box\,\Box\,\ldots$ translates to
$a(b(a(a(\triangleright))))$
Left of the reading head: $\ldots\,\Box\,\Box\,a\,b\,a\,a$ translates to $a(a(b(a(\triangleright))))$

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Left of the reading head: $\ldots\,\Box\,\Box\,a\,b\,a\,a$ translates to $a(a(b(a(\triangleright))))$
Tape content $\ldots\,\Box\,w\,\underline{a}\,v\Box\,\ldots$ in state q becomes $q(w^R, a(v))$
(q is reading a, the first symbol of $a\,v$)

Encoding a Turing Machine M as a TRS R_M

Translating the transition function δ:

$$
\begin{array}{rcllcl}
q(x, f(y)) & \longrightarrow & q'(f'(x), y) & \text{if} & \delta(q, f) & = & (q', f', R) \\
q(g(x), f(y)) & \longrightarrow & q'(x, g(f'(y))) & \text{if} & \delta(q, f) & = & (q', f', L)
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\end{array}
$$

And special rewrite rules for dealing with the left-/rightmost blank:

$$
\begin{array}{rclcrcl}
q(\triangleright, f(y)) & \longrightarrow & q'(\triangleright, \Box(f'(y))) & \text{if} & \delta(q, f) & = & (q', f', L) \\
q(x, \triangleright) & \longrightarrow & q'(f'(x), \triangleright) & \text{if} & \delta(q, \Box) & = & (q', f', R) \\
q(g(x), \triangleright) & \longrightarrow & q'(x, g(f'(\triangleright))) & \text{if} & \delta(q, \Box) & = & (q', f', L) \\
q(\triangleright, \triangleright) & \longrightarrow & q'(\triangleright, \Box(f'(\triangleright))) & \text{if} & \delta(q, \Box) & = & (q', f', L)
\end{array}
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Σ^0_1-completeness of WCR

WCR is in Σ^0_1: By the Critical Pairs Lemma, WCR_R holds if and only if all critical pairs of R are convergent.
A Turing machine can compute on the input of a TRS R all (finitely many) critical pairs, and on the input of a TRS R and a term t all (finitely many) one step reducts of t.

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A Turing machine can compute on the input of a TRS R all (finitely many) critical pairs, and on the input of a TRS R and a term t all (finitely many) one step reducts of t.
So it suffices to show that the following is in Σ^0_1:

> *Decide on the input of a TRS S, $n \in \mathbb{N}$ and terms $t_1, s_1, \ldots, t_n, s_n$ whether for every $i = 1, \ldots, n$ the terms t_i and s_i have a common reduct.*

This property can easily be described by a Σ^0_1 formula.

Σ_1^0-completeness of WCR

WCR is Σ_1^0-hard: We define TRS S to consist of the rules of R_{M} extended by the following:

$$\mathsf{run} \to \mathsf{T} \qquad \mathsf{run} \to q_0(\triangleright, \triangleright)$$
$$q(x, f(y)) \to \mathsf{T} \quad \text{for every } f \in \Gamma \text{ such that } \delta(q, f) \text{ is undefined} .$$

The only critical pair is $\mathsf{T} \leftarrow \mathsf{run} \to q_0(\triangleright, \triangleright)$. We have:

$$q_0(\triangleright, \triangleright) \to_S^* \mathsf{T} \text{ if and only if M halts on the blank tape.}$$

So:

$$\mathsf{WCR}(S) \text{ if and only if M halts on the blank tape.}$$

Π^0_2-completeness of CR

CR is in Π^0_2:

$$
\begin{aligned}
\mathrm{CR}_R \iff\ & \forall t \in \mathbb{N}.\ \forall r_1, r_2 \in \mathbb{N}.\ \exists r_1', r_2' \in \mathbb{N}. \\
& (((t \text{ is a term}) \text{ and } (r_1,\ r_2 \text{ are reductions}) \\
& \quad \text{and } t \equiv \mathit{first}(r_1) \equiv \mathit{first}(r_2)) \\
& \quad \Rightarrow ((r_1' \text{ and } r_2' \text{ are reductions}) \\
& \qquad \text{and } (\mathit{last}(r_1) \equiv \mathit{first}(r_1')) \text{ and } (\mathit{last}(r_2) \equiv \mathit{first}(r_2'))\ . \\
& \qquad \text{and } (\mathit{last}(r_1') \equiv \mathit{last}(r_2'))))
\end{aligned}
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We change the TRS R_M in such a way that

$$M \text{ halts on all inputs} \iff R_M \text{ is CR}$$

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Idea: use an extension of R_M with the following rules:

$$\begin{aligned} \mathsf{run}(x,y) &\to \mathsf{T} \\ \mathsf{run}(x,y) &\to q_0(x,y) \\ q(x,f(y)) &\to \mathsf{T} \qquad\qquad \text{for every } f \in \Gamma \text{ with } \delta(q,f) \text{ undefined} \end{aligned}$$

Then it seems that

$$\mathrm{CR}(R_M^+) \iff \text{ the Turing machine M halts on all configurations.}$$

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Then it seems that

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However, we only have $\Longrightarrow$. With $\Longleftarrow$ a problem arises if s and t contain variables.

Π^0_2-hardness of CR

For a Turing machines M we define the TRS S_{M} as R_M extended with

$$\mathsf{run}(x, \triangleright) \to \mathsf{T} \tag{1}$$

$$\mathsf{run}(\triangleright, y) \to q_0(\triangleright, y) \tag{2}$$

$$q(x, f(y)) \to \mathsf{T} \qquad \text{if } \delta(q, f) \text{ undefined} \tag{3}$$

$$\mathsf{run}(x, \mathsf{S}(y)) \to \mathsf{run}(\mathsf{S}(x), y) \tag{4}$$

$$\mathsf{run}(\mathsf{S}(x), y) \to \mathsf{run}(x, \mathsf{S}(y)) \,. \tag{5}$$

Π^0_2-hardness of CR

For a Turing machines M we define the TRS S_M as R_M extended with

$$\begin{aligned}
\mathsf{run}(x, \triangleright) &\to \mathsf{T} && && (1)\\
\mathsf{run}(\triangleright, y) &\to q_0(\triangleright, y) && && (2)\\
q(x, f(y)) &\to \mathsf{T} && \text{if } \delta(q, f) \text{ undefined} && (3)\\
\mathsf{run}(x, \mathsf{S}(y)) &\to \mathsf{run}(\mathsf{S}(x), y) && && (4)\\
\mathsf{run}(\mathsf{S}(x), y) &\to \mathsf{run}(x, \mathsf{S}(y))\,. && && (5)
\end{aligned}$$

Then the only cause for non-confluence can be (t_1, t_2 are ground terms)

$$q_0(\triangleright, s_1) \leftarrow_{(2)} \mathsf{run}(s_1, \triangleright) \leftarrow^*_{(4)} \mathsf{run}(t_1, t_2) \to^*_{(5)} \mathsf{run}(s_1, \triangleright) \to_{(1)} \mathsf{T}$$

Thus we can prove

$$\mathsf{CR}(S_M) \iff \text{the Turing machine M halts on all inputs.}$$

Dependency Pair problems for TRSs

- For relations $\rightarrow_R, \rightarrow_S$ we write $\rightarrow_R / \rightarrow_S$ for $\rightarrow_S^* \cdot \rightarrow_R$.
- $\rightarrow_{R,\epsilon}$ denotes R-reduction,but only at the top of a term.
- Write $\mathrm{SN}(R_{\mathrm{top}}/S)$ instead of $\mathrm{SN}(\rightarrow_{R,\epsilon}/\rightarrow_S)$.

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$\mathrm{SN}(R_{\mathrm{top}}/S)$ is the finiteness of the dependency pair problem for $\{R, S\}$.
So $\mathrm{SN}(R_{\mathrm{top}}/S)$ means that every infinite $\rightarrow_{R,\epsilon} \cup \rightarrow_S$ reduction, contains only finitely many $\rightarrow_{R,\epsilon}$ steps.

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So $\mathrm{SN}(R_{\mathrm{top}}/S)$ means that every infinite $\to_{R,\epsilon} \cup \to_S$ reduction, contains only finitely many $\to_{R,\epsilon}$ steps.
Motivation: There a simple syntactic construction DP such that for any TRS S we have

$$\mathrm{SN}(\mathrm{DP}(S)_{\mathrm{top}}/S) \iff \mathrm{SN}(S).$$

Dependency pair problems

The dependency pair problem $\{R, S\}$ is finite if $\mathsf{SN}(R_{\mathsf{top}}/S)$.

$$\mathsf{SN}(R_{\mathsf{top}}/S) \;:=\; \to_S^* \cdot \to_{R,\epsilon} \text{ is SN}$$

This seems a "standard" SN-for-TRS problem, so should be Π_2^0 ...

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The dependency pair problem $\{R, S\}$ is finite if $\mathsf{SN}(R_{\mathsf{top}}/S)$.

$$\mathsf{SN}(R_{\mathsf{top}}/S) \;:=\; \rightarrow_S^* \cdot \rightarrow_{R,\epsilon} \;\text{ is SN}$$

This seems a "standard" SN-for-TRS problem, so should be Π_2^0 ...
But: $\rightarrow_S^* \cdot \rightarrow_{R,\epsilon}$ is not finitely branching.

Example

$f(x) \longrightarrow_S g(f(x))$
$g(x) \longrightarrow_R a$
Finite DP problem, but $\rightarrow_S^* \cdot \rightarrow_{R,\epsilon}$ is not finitely branching:
$f(x) \rightarrow_S^* g^n((f(x)) \rightarrow_{R,\epsilon} a$.

SN for non-finitely branching systems (ARSs)

$$\mathsf{SN}_R(a) := \forall \alpha : \mathbb{N} \to \mathbb{N}\, (\alpha(0) = a \implies \exists i\, \neg(\alpha(i) \longrightarrow_R \alpha(i+1)))$$

"There is no infinite reduction starting from *a*".
This is a Π^1_1-statement, so finiteness of DP problems is in the class Π^1_1.

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Is it Π^1_1-complete?
Yes: we prove

$$\mathsf{WF}(>_M) \Longleftrightarrow \mathsf{SN}(S^M_{\mathsf{top}}/S^M)$$

for a suitable S_M constructed from M. This reduces $\mathsf{WF}(>_M)$ to $\mathsf{SN}(S^M_{\mathsf{top}}/S^M)$, thus showing Π^1_1-hardness of dependency pair problems.

DP is Π^1_1-complete

We now reduce well-foundedness of $>_M$ to $\mathrm{SN}(S^M_{\mathrm{top}}/S^M)$ and thus obtain that DP is Π^1_1-complete.

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IDEA: We define a TRS S^M such that

S^M has an infinite reduction iff $\neg\mathrm{WF}(>_M)$,

and this reduction "keeps coming back to the top level".

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IDEA: We define a TRS S^M such that
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and this reduction "keeps coming back to the top level".
We want to mimick a computation that
1. arbitrarily picks a number n_1
2. arbitrarily picks a number n_2
3. checks if $n_1 >_M n_2$, if "no" stops, if "yes" replaces n_1 by n_2 and continues with (2)

Notation: we write $\overline{n}$ to denote $S^n(0(\triangleright))$

DP is Π^1_1-complete

First we add

$$q(x, 0(y)) \quad \longrightarrow \quad \mathsf{T} \quad \text{if} \quad \delta(q, 0) = \text{undefined}$$

so that we have

$$n >_M p \text{ iff } q_0(\overline{n}, \overline{p}) \rightarrow^*_R \mathsf{T}$$

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Then (but this is too simple ...): to pick arbitrary numbers we introduce the following TRS

$$\begin{aligned} \text{pick} &\longrightarrow \mathsf{S}(\text{pick}) \\ \text{pick} &\longrightarrow 0(\triangleright) \end{aligned}$$

and we add

$$\text{try}(\mathsf{T}, x, y) \longrightarrow \text{try}(q(x, y), y, \text{pick})$$

DP is Π^1_1-complete

The intention is to have

$$\begin{array}{rll} \mathsf{try}(\mathsf{T}, \mathsf{pick}, \mathsf{pick}) & \to^* & \\ \mathsf{try}(\mathsf{T}, \overline{n}_1, \overline{n}_2) & \to^* & \mathsf{try}(q_0(\overline{n}_1, \overline{n}_2), \overline{n}_2, \mathsf{pick}) \longrightarrow \\ \mathsf{try}(\mathsf{T}, \overline{n}_2, \overline{n}_3) & \to^* & \mathsf{try}(q_0(\overline{n}_2, \overline{n}_3), \overline{n}_3, \mathsf{pick}) \longrightarrow \ldots \end{array}$$

only if there is an infinite descending sequence $n_1 >_M n_2 >_M n_3 \ldots$

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\mathrm{try}(\mathsf{T}, \overline{n}_2, \overline{n}_3) &\rightarrow^* \quad \mathrm{try}(q_0(\overline{n}_2, \overline{n}_3), \overline{n}_3, \mathrm{pick}) \longrightarrow \dots
\end{aligned}$$

only if there is an infinite descending sequence $n_1 >_M n_2 >_M n_3 \dots$
However we also have:

$$\begin{aligned}
\mathrm{try}(\mathsf{T}, \mathrm{pick}, \mathrm{pick}) &\rightarrow^* \\
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\mathrm{try}(\mathsf{T}, \mathrm{pick}, \mathrm{pick}) &\rightarrow^* \quad \dots
\end{aligned}$$

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if $n_1 >_M n_2$
Problem: $\mathrm{try}(\mathsf{T}, u, s)$ should only reduce if u and s represent a number.

DP is Π^1_1-complete

To pick arbitrary numbers we introduce the following TRS

$$\begin{array}{rcl} \mathsf{pick} & \longrightarrow & c(\mathsf{pick}) \\ \mathsf{pick} & \longrightarrow & \mathsf{ok}(0(\triangleright)) \\ c(\mathsf{ok}(x)) & \longrightarrow & \mathsf{ok}(\mathsf{S}(x)) \end{array}$$

Then $\mathsf{pick} \to^* c^n(\mathsf{pick}) \longrightarrow c^n(\mathsf{ok}(0(\triangleright))) \longrightarrow \mathsf{ok}(\mathsf{S}^n(0(\triangleright))) \equiv \overline{n}$

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Lemma $\mathsf{pick} \rightarrow^* \mathsf{ok}(t) \iff \exists n(t = \mathsf{S}^n(0(\triangleright)))$

DP is Π^1_1-complete

Finally we add the following rewrite rule

$$\mathsf{try}(\mathsf{T}, \mathsf{ok}(x), \mathsf{ok}(y)) \longrightarrow \mathsf{try}(q_0(x, y), \mathsf{ok}(y), \mathsf{pick})$$

Then: the term $\mathsf{try}(\mathsf{T}, \mathsf{pick}, \mathsf{pick})$ is $\mathsf{SN}(R_{\mathrm{top}}/S)$ iff $>_M$ is well-founded.

Proof: The only infinite reduction that is possible is of the form

$$\begin{array}{rcl} \mathsf{try}(\mathsf{T}, \mathsf{pick}, \mathsf{pick}) & \to^* & \\ \mathsf{try}(\mathsf{T}, \mathsf{ok}(\overline{n}_1), \mathsf{ok}(\overline{n}_2)) & \to^* & \mathsf{try}(q_0(\overline{n}_1, \overline{n}_2), \mathsf{ok}(\overline{n}_2), \mathsf{pick}) \longrightarrow \\ \mathsf{try}(\mathsf{T}, \mathsf{ok}(\overline{n}_2), \mathsf{ok}(\overline{n}_3)) & \to^* & \mathsf{try}(q_0(\overline{n}_2, \overline{n}_3), \mathsf{ok}(\overline{n}_3), \mathsf{pick}) \longrightarrow \\ & \ldots & \end{array}$$

if $n_1 >_M n_2 >_M n_3 \ldots$

Remarks / Conclusions / Future work

Remarks

- In $\mathrm{DP}^{\mathrm{min}}$, we restrict $\to_S^* \cdot \to_{R,\epsilon}$ to terms that are $\mathrm{SN}(S)$. $\mathrm{DP}^{\mathrm{min}}$ is Π_2^0-complete (see paper).
- $\mathrm{SN}^\omega(R)$ is Π_1^1-complete (see paper).

Future work:

- Characterize "all" properties of TRSs, distinguishing between "ground terms" and "all terms": UN, ...
- Characterize $\mathrm{WN}^\omega(R)$.
  $\mathrm{WN}^\omega(R) := \forall t \exists \alpha (\ldots) \iff \exists \alpha \forall t (\ldots) \in \Pi_1^1$.
- Extend to infinite terms. $\mathrm{SN}_\infty^\omega(R) := \forall \beta \forall \alpha (\ldots) \in \Pi_1^1$.
  $\mathrm{WN}_\infty^\omega(R) := \forall \beta \exists \alpha (\ldots) \in \Sigma_2^1$.
- Make the generalizations to SN^∞, WN^∞ precise, for reduction of all countable ordinal length.
- Study the proof-theoretic complexity of productivity