Deriving natural deduction rules from truth tables (Extended version)

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Abstract

We develop a general method for deriving natural deduction rules from the truth table for a connective. The method applies to both constructive and classical logic. This implies we can derive "constructively valid" rules for any (classical) connective. We show this constructive validity by giving a general Kripke semantics, that is shown to be sound and complete for the constructive rules. For the well-known connectives, like $\vee$, $\wedge$, $\rightarrow$, the constructive rules we derive are equivalent to the natural deduction rules we know from Gentzen and Prawitz. However, they have a different shape, because we want all our rules to have a standard "format", to make it easier to define the notions of cut and to study proof reductions. In style they are close to the "general elimination rules" by Von Plato [14]. The rules also shed some new light on the classical connectives: e.g. the classical rules we derive for $\rightarrow$ allow to prove Peirce's law. Our method also allows to derive rules for connectives that are usually not treated in natural deduction textbooks, like the "if-then-else", whose truth table is clear but whose constructive deduction rules are not. We prove that "if-then-else", in combination with $\perp$ and $\top$, is functionally complete (all other constructive connectives can be defined from it). We define the notion of cut, generally for any constructive connective and we describe the process of "cut-elimination". Following the Curry-Howard isomorphism, we can give terms to deductions and we study cut-elimination as term reduction. We prove that reduction is strongly normalizing for constructive if-then-else logic.

Keywords and phrases constructive logic, natural deduction, cut-elimination, Kripke semantics, Curry-Howard isomorphism

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1 Introduction

Natural deduction rules come in various forms, where the tree format is the most well-known, where one either puts formulas A as the nodes and leaves of the tree, or sequents $\Gamma \vdash A$, where Γ is a sequence or a finite set of formulas. Other formalisms use a linear format, using flags or boxes to explicitly manage the open and discharged assumptions.

We use a tree format with sequents in the nodes and leaves, where all rules have a special form:

$$\frac{\Gamma \vdash \Phi_1 \quad \dots \quad \Gamma \vdash \Phi_n \qquad \Gamma, \Psi_1 \vdash D \quad \dots \quad \Gamma, \Psi_m \vdash D}{\Gamma \vdash D}$$

So if the conclusion of a rule is $\Gamma \vdash D$, then the hypotheses of the rule can be of one of two forms:

1. $\Gamma, \Psi \vdash D$: we still need to prove D from Γ, but we are now also allowed to use Ψ as additional assumption. We call Ψ a Casus.
2. $\Gamma \vdash \Phi$: in stead of proving D from Γ, we now need to prove Φ from Γ. We call Φ a Lemma.

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One obvious advantage is that we don't have to give the Γ explicitly, as it can be retrieved from the other information in a deduction. So, we will present the deduction rules without the Γ in the format

$$\frac{\vdash \Phi_1 \quad \dots \quad \vdash \Phi_n \qquad \Psi_1 \vdash D \quad \dots \quad \Psi_m \vdash D}{\vdash D}$$

For every connective we have elimination rules and introduction rules, where the introduction rules come in a intuitionistic and a classical variant. The elimination rules have the following form, where we indicate occurrences of Casus by C and occurrences of Lemma by L. One of the occurrences of a Lemma is the formula we eliminate, which we indicate by E. (So Φ^E below is a special case of a Lemma.)

$$\frac{\vdash \Phi^E \quad \vdash \Phi_1^L \quad \dots \quad \vdash \Phi_n^L \qquad \Psi_1^C \vdash D \quad \dots \quad \Psi_m^C \vdash D}{\vdash D}\,\mathrm{el}$$

The introduction rules have a classical and an intuitionistic form; the following form is the classical one, where we again indicate occurrences of Casus by C and occurrences of Lemma by L. One of the occurrences of a Casus is the formula we "introduce", which we indicate by I. (So Φ^I below is a special case of a Casus.) The duality between elimination and introduction is clearly visible from these rules.

$$\frac{\Phi^I \vdash D \quad \vdash \Phi_1^L \quad \dots \quad \vdash \Phi_n^L \qquad \Psi_1^C \vdash D \quad \dots \quad \Psi_m^C \vdash D}{\vdash D}\,\mathrm{in}^c$$

The intuitionistic introduction rules have the following form

$$\frac{\vdash \Phi_1^L \quad \dots \quad \vdash \Phi_n^L \qquad \Psi_1^C \vdash \Phi \quad \dots \quad \Psi_m^C \vdash \Phi}{\vdash \Phi}\,\mathrm{in}^i$$

We see that, compared to the classical rule, the D has been replaced by Φ, the formula we introduce, and we have omitted the first premise, which is $\Phi^I \vdash \Phi$, because it is trivial. We extract these rules from a truth table as described in the following Definition.

▶ **Definition 1.** Suppose we have an n-ary connective c with a truth table t_c (with 2^n rows). We write $\varphi = c(p_1, \dots, p_n)$, where $p_1, \dots, p_n$ are proposition letters and we write $\Phi = c(A_1, \dots, A_n)$, where $A_1, \dots, A_n$ are arbitrary propositions. Each row of t_c gives rise to an elimination rule or an introduction rule for c in the following way.

$$\begin{array}{ccc|c} p_1 & \dots & p_n & \varphi \\ \hline a_1 & \dots & a_n & 0 \end{array} \quad \mapsto \quad \frac{\vdash \Phi \quad \dots \vdash A_j \text{ (if } a_j = 1) \dots \quad \dots A_i \vdash D \text{ (if } a_i = 0) \dots}{\vdash D}\,\mathrm{el}$$

$$\begin{array}{ccc|c} p_1 & \dots & p_n & \varphi \\ \hline b_1 & \dots & b_n & 1 \end{array} \quad \mapsto \quad \frac{\dots \vdash A_j \text{ (if } b_j = 1) \dots \quad \dots A_i \vdash \Phi \text{ (if } b_i = 0) \dots}{\vdash \Phi}\,\mathrm{in}^i$$

$$\begin{array}{ccc|c} p_1 & \dots & p_n & \varphi \\ \hline c_1 & \dots & c_n & 1 \end{array} \quad \mapsto \quad \frac{\Phi \vdash D \quad \dots \vdash A_j \text{ (if } c_j = 1) \dots \quad \dots A_i \vdash D \text{ (if } c_i = 0) \dots}{\vdash D}\,\mathrm{in}^c$$

If $a_j = 1$ in t_c, then A_j occurs as a Lemma in the rule; if $a_i = 0$ in t_c, then A_i occurs as a Casus. The rules are given in abbreviated form and it should be understood that all judgments can be used with an extended hypotheses set Γ. So the elimination rule in full reads as follows (where Γ is a set of propositions).

$$\frac{\Gamma \vdash \Phi \quad \dots \Gamma \vdash A_j \text{ (if } a_j = 1) \dots \quad \dots \Gamma, A_i \vdash D \text{ (if } a_i = 0) \dots}{\Gamma \vdash D}\,\mathrm{el}$$

▶ **Definition 2.** Given a set of connectives $\mathcal{C} := \{c_1, \dots, c_n\}$, we define the *intuitionistic* and *classical* natural deduction systems for $\mathcal{C}$, $\mathsf{IPC}_{\mathcal{C}}$ and $\mathsf{CPC}_{\mathcal{C}}$ as follows.

- Both $\mathsf{IPC}_{\mathcal{C}}$ and $\mathsf{CPC}_{\mathcal{C}}$ have an *axiom rule*

$$\frac{}{\Gamma \vdash A}\ \text{axiom(if } A \in \Gamma)$$

- $\mathsf{IPC}_{\mathcal{C}}$ has the elimination rules for the connectives in $\mathcal{C}$ and the intuitionistic introduction rules for the connectives in $\mathcal{C}$, as defined in Definition 1.
- $\mathsf{CPC}_{\mathcal{C}}$ has the elimination rules for the connectives in $\mathcal{C}$ and the classical introduction rules for the connectives in $\mathcal{C}$, as defined in Definition 1.

We write $\Gamma \vdash_i A$ if $\Gamma \vdash A$ is derivable using the derivation rules of $\mathsf{IPC}_{\mathcal{C}}$. We write $\Gamma \vdash_c A$ if $\Gamma \vdash A$ is derivable using the derivation rules of $\mathsf{CPC}_{\mathcal{C}}$.

▶ **Example 3.** From the truth table we derive the following intuitionistic rules for $\wedge$, 3 elimination rules and one introduction rule:

$$\frac{\vdash A \wedge B \quad A \vdash D \quad B \vdash D}{\vdash D}\ \wedge\text{-el}_a \qquad \frac{\vdash A \wedge B \quad A \vdash D \quad \vdash B}{\vdash D}\ \wedge\text{-el}_b$$

$$\frac{\vdash A \wedge B \quad \vdash A \quad B \vdash D}{\vdash D}\ \wedge\text{-el}_c \qquad \frac{\vdash A \quad \vdash B}{\vdash A \wedge B}\ \wedge\text{-in}$$

These rules are all intuitionistically correct, as one can observe by inspection. We will show that these are equivalent to the well-known intuitionistic rules. We will also show how these rules can be optimized and be reduced to 2 elimination rules and 1 introduction rule.

From the truth table we also derive the following rules for $\neg$, 1 elimination rule and 1 introduction rule, a classical and an intuitionistic one.

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D}\ \neg\text{-el} \qquad \frac{A \vdash \neg A}{\vdash \neg A}\ \neg\text{-in}^i \qquad \frac{\neg A \vdash D \quad A \vdash D}{\vdash D}\ \neg\text{-in}^c$$

As an example of the classical derivation rules we show that $\neg\neg A \vdash A$ is derivable:

$$\frac{\dfrac{\neg\neg A, \neg A \vdash \neg\neg A \quad \neg\neg A, \neg A \vdash \neg A}{\neg\neg A, \neg A \vdash A}\ \neg\text{-el} \qquad \neg\neg A, A \vdash A}{\neg\neg A \vdash A}\ \neg\text{-in}^c$$

It can be proven that $\neg\neg A \vdash A$ is not derivable with the intuitionistic rules. As an example of the intuitionistic derivation rules we show that $A \vdash \neg\neg A$ is derivable:

$$\frac{\dfrac{A, \neg A \vdash \neg A \quad A, \neg A \vdash A}{A, \neg A \vdash \neg\neg A}\ \neg\text{-el}}{A \vdash \neg\neg A}\ \neg\text{-in}^i$$

In the intuitionistic case, there is an obvious notion of *cut* that we study: an intro of Φ immediately followed by an elimination of Φ. In such case there is at least one k for which $a_k \neq b_k$. In case $a_k = 0, b_k = 1$, we have a sub-derivation Σ of $\vdash \Phi_k$ and a sub-derivation Θ of $\Phi_k \vdash D$ and we can "plug" Σ on top of Θ to obtain a derivation of $\vdash D$. In case $a_k = 1, b_k = 0$, we have a sub-derivation Σ of $\Phi_k \vdash \Phi$ and a sub-derivation Θ of $\vdash \Phi_k$ and we can "plug" Θ on top of Σ to obtain a derivation of $\vdash \Phi$. This is then used as a hypothesis for the elimination rule (that remains in this case) in stead of the original one that was a consequence of the introduction rule (that now disappears). Note that in general there are more such k, so the general cut-elimination procedure is non-deterministic. We view this non-determinism as a natural feature in natural deduction; the fact that for some connectives (or combination of connectives), cut-elimination is deterministic is an "emerging" property.

1.1 Contribution of the paper and related work

Natural deduction has been studied extensively, since the original work by Gentzen, both for classical and intuitionistic logic. Overviews can be found in [13] and [8]. Also the generalization of natural deduction to include other connectives or allow different derivation rules has been studied by various researchers. Notably, there is the work of Schroeder-Heister [11], Von Plato [14], Milne [7] and Francez and Dyckhoff [5, 4] is related to ours. Schroeder-Heister studies general formats of natural deduction where also rules may be discharged (as opposed to the normal situation where only formulas may be discharged). He also studies a general rule format for intuitionistic logic and shows that the connectives $\wedge, \vee, \rightarrow, \perp$ are complete for it. Von Plato, Milne, Francez and Dyckhoff study "generalized elimination rules", where the idea is that elimination rules arise naturally from the introduction rules, following Prawitz's [10] inversion principle: "the conclusion obtained by an elimination does not state anything more than what must have already been obtained if the major premiss of the elimination was inferred by an introduction".

A difference is that we focus not so much on the rules but on the fact that we can define different and new connectives constructively. In our work, we do not take the introduction rules as primary, with the elimination rules defined from them, but we derive elimination and introduction rules directly from the truth table. Then we optimize them, which can be done in various ways, where we adhere to a fixed format for the rules. Many of the generalized elimination rules, for example for $\wedge$, appear naturally as a consequence of our approach. of deriving the rules from the truth table.

The idea of deriving deduction rules from the truth table also occurs in the work of Milne [7], but in a slightly different way: from the introduction rules, a truth table is derived and then the classical elimination rules are derived from the truth table. For the if-then-else connective, this amounts to rules equivalent to ours (see Appendix 8), but not optimized. We start from the truth table and also derive rules for constructive logic.

In Section 3 we give a complete Kripke semantics for the constructive connectives. This is reminiscent of the *Lindenbaum lemma* that is used in [7] to prove classical completeness. The Kripke semantics also allows us to prove some meta properties about the rules. For example, we give a generalization of the *disjunction property* in intuitionistic logic. In Section 4 we define and study cuts precisely, for the intuitionistic case. We look more in detail into the logic with just if-then-else and we prove that cut-elimination is strongly normalizing by studying the reduction of proof terms.

2 Simple properties and examples

We first define precisely how the "plugging one derivation in another" works.

▶ **Lemma 4.** *If* $\Gamma \vdash \varphi$ *and* $\Delta, \varphi \vdash \psi$*, then* $\Gamma, \Delta \vdash \psi$

Proof. By a simple induction on the derivation of $\Delta, \varphi \vdash \psi$, using the fact that, in general (for all Γ, Γ' and φ): If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \varphi$. ◀

We can be a bit more precise about what is happening in the proof of Lemma 4. If Π is the derivation of $\Delta, \varphi \vdash \psi$, due to the format of our rules, the only place in Π where the hypothesis φ can be used is at a leaf of Π, in an instance of the (axiom) rule. These leaves are of the shape $\Delta', \varphi \vdash \varphi$ for some $\Delta' \supseteq \Delta$.

If Σ is the derivation of $\Gamma \vdash \varphi$, then Σ is also a derivation of $\Delta', \Gamma \vdash \varphi$ (for any Δ). So, we can replace each leaf of Π that is an instance of an axiom $\Delta', \varphi \vdash \varphi$ by a derivation Σ of $\Delta', \Gamma \vdash \varphi$, to obtain a derivation of $\Gamma, \Delta \vdash \psi$. We introduce some notation to support this.

▶ Notation 5. If Σ is a derivation of $\Gamma \vdash \varphi$ and Π is a derivation of $\Delta, \varphi \vdash \psi$, then we have a derivation of $\Gamma, \Delta \vdash \psi$ that looks like this:

$$\begin{array}{c} \begin{array}{ccc} \vdots\,\Sigma & & \vdots\,\Sigma \\ \Gamma \vdash \varphi & \ldots & \Gamma \vdash \varphi \end{array} \\ \vdots\,\Pi \\ \Delta \vdash \psi \end{array}$$

So in Π, every application of an (axiom) rule at a leaf, deriving $\Delta' \vdash \varphi$ for some $\Delta' \supseteq \Delta$ is replaced by a copy of a derivation Σ, which is also a derivation of $\Delta', \Gamma \vdash \varphi$.

In Definitions 1 and 2, we have given the precise rules for our logic, in intuitionistic and classical format. We can freely reuse formulas and weaken the context, so the structural rules of contraction and weakening are wired into the system. In examples, to simplify derivations we will often use the following format for an elimination rule (and equivalently for an introduction rule).

$$\frac{\Gamma_0 \vdash \Phi \quad \ldots \Gamma_j \vdash A_j \text{ (if } a_j = 1) \ldots \quad \ldots \Gamma_i, A_i \vdash D \text{ (if } a_i = 0) \ldots}{\cup_{k=0}^{n} \Gamma_k \vdash D}\,\text{el}$$

To reduce the number of rules, we can take a number of rules together and drop one or more hypotheses. We show this by again looking at the example of the rules for $\wedge$ (Example 3).

▶ **Example 6.** From the truth table we have derived the following 3 intuitionistic elimination rules:

$$\frac{\vdash A \wedge B \quad A \vdash D \quad B \vdash D}{\vdash D}\,\wedge\text{-el}_a \qquad \frac{\vdash A \wedge B \quad A \vdash D \quad \vdash B}{\vdash D}\,\wedge\text{-el}_b \qquad \frac{\vdash A \wedge B \quad \vdash A \quad B \vdash D}{\vdash D}\,\wedge\text{-el}_c$$

These rules can be reduced to the following 2 equivalent elimination rules:

$$\frac{\vdash A \wedge B \quad A \vdash D}{\vdash D}\,\wedge\text{-el}_1 \qquad \frac{\vdash A \wedge B \quad B \vdash D}{\vdash D}\,\wedge\text{-el}_2$$

It can be shown that these sets of rules are equivalent (and equivalent again to the more standard $\wedge$-elimination rules that are known as "first and second projection"). Here we only show the derivability of the new rules from the standard ones (the ones we have derived from the truth table), by giving a derivation for the $\wedge$-el$_1$ rule. (The $\wedge$-el$_2$ rule is similar.) Suppose we have derivations of $\Gamma \vdash A \wedge B$ and of $\Gamma, A \vdash D$. Then we have the following derivation, using the rules $\wedge$-el$_a$ and $\wedge$-el$_b$:

$$\frac{\Gamma \vdash A \wedge B \quad \Gamma, A \vdash D \quad \dfrac{\Gamma, B \vdash A \wedge B \quad \Gamma, B, A \vdash D \quad \Gamma, B \vdash B}{\Gamma, B \vdash D}}{\Gamma \vdash D}$$

The general method here is that we can replace two rules that only differ in one hypothesis, which in one rule occurs as a Lemma and in the other as a Casus, by one rule where the hypothesis is removed. It will be clear that the Γ's above are not relevant for the argument, so we will not write these.

▶ **Lemma 7.** *A system with two derivation rules of the form*

$$\frac{\vdash \Phi_1 \ \ldots \ \vdash \Phi_n \quad \Psi_1 \vdash D \ \ldots \ \Psi_m \vdash D \quad A \vdash D}{\vdash D} \qquad \frac{\vdash \Phi_1 \ \ldots \ \vdash \Phi_n \quad \vdash A \quad \Psi_1 \vdash D \ \ldots \ \Psi_m \vdash D}{\vdash D}$$

is equivalent to the system with these two rules replaced by

$$\frac{\vdash \Phi_1 \ \ldots \ \vdash \Phi_n \quad \Psi_1 \vdash D \ \ldots \ \Psi_m \vdash D}{\vdash D}$$

Proof. The implication from bottom to top is immediate. From top to bottom, suppose we have the two given rules. We now derive the bottom one. Assume we have derivations of $\vdash \Phi_1, \ldots, \vdash \Phi_n, \Psi_1 \vdash D, \ldots, \Psi_m \vdash D$. We now have the following derivation of $\vdash D$.

$$\frac{\vdash \Phi_1 \ \ldots \ \vdash \Phi_n \quad \Psi_1 \vdash D \ \ldots \ \Psi_m \vdash D \qquad \dfrac{A \vdash \Phi_1 \ \ldots \ A \vdash \Phi_n \quad A \vdash A \quad A, \Psi_1 \vdash D \ \ldots \ A, \Psi_m \vdash D}{A \vdash D}}{\vdash D}$$

◀

Similarly, we can replace a rule which has only one Casus by a rule where the Casus is the conclusion. To illustrate this: the simplified elimination rules for $\wedge$, $\wedge$-el$_1$ and $\wedge$-el$_2$ have only one Casus. The rule $\wedge$-el$_1$ (left) can thus be replaced by the rule $\wedge$-el$'_1$ (right), which is the usual projection rule.

$$\frac{\vdash A \wedge B \quad A \vdash D}{\vdash D}\ \wedge\text{-el}_1 \qquad \frac{\vdash A \wedge B}{\vdash A}\ \wedge\text{-el}'_1$$

There is a general Lemma stating this simplification is correct.

▶ **Lemma 8.** *A system with a derivation rule of the form to the left is equivalent to the system with this rule replaced by the rule on the right.*

$$\frac{\vdash \Phi_1 \ \ldots \ \vdash \Phi_n \quad \Psi \vdash D}{\vdash D} \qquad\qquad \frac{\vdash \Phi_1 \ \ldots \ \vdash \Phi_n}{\vdash \Psi}$$

Proof. The implication from left to right is immediate. From right to left, assume we have derivations of $\vdash \Phi_1, \ldots, \vdash \Phi_n$. Then, by the rule to the right, we have $\Gamma \vdash \Psi$. Now assume we also have a derivation of $\Psi \vdash D$. By Lemma 4, we also have a derivation of $\Gamma \vdash D$.

◀

2.1 The connective if-then-else

If we look at if-then-else, which has the obvious (classical) truth table semantics as a ternary connective (see also the Appendix 8), and we apply the optimizations of Lemmas 7 and 8 we obtain the following intuitionistic rules, where we write $A{\to}B/C$ for if A then B else C.

$$\frac{\vdash A{\to}B/C \quad \vdash A}{\vdash B}\ \text{then-el} \qquad \frac{\vdash A{\to}B/C \quad A \vdash D \quad C \vdash D}{\vdash D}\ \text{else-el}$$

$$\frac{\vdash A \quad \vdash B}{\vdash A{\to}B/C}\ \text{then-in} \qquad \frac{A \vdash A{\to}B/C \quad \vdash C}{\vdash A{\to}B/C}\ \text{else-in}$$

Basically, $A{\to}B/C$ is equivalent to $(A \to B) \wedge (A \vee C)$. It can be shown that $A{\to}B/C$ is "in between" other constructive renderings of if-then-else:

$$(A \wedge B) \vee (\neg A \wedge C) \quad \overset{\not\dashv}{\vdash} \quad A{\to}B/C \quad \overset{\not\dashv}{\vdash} \quad (A \to B) \wedge (\neg A \to C)$$

The left-to-right can easily be derived, for the non-derivability of the reverse, we need a Kripke model (see Section 3).

If we compare with well-known classical rules for if-then-else, we observe that two of them hold, while the other fails.

▶ Fact 9. **1.** $\mathsf{if}\ A\ \mathsf{then}\ B\ \mathsf{else}\ B \vdash B$ and $B \vdash \mathsf{if}\ A\ \mathsf{then}\ B\ \mathsf{else}\ B$,

2. $\mathsf{if}\ (\mathsf{if}\ A\ \mathsf{then}\ B\ \mathsf{else}\ C)\ \mathsf{then}\ D\ \mathsf{else}\ E \nvdash \mathsf{if}\ A\ \mathsf{then}\ (\mathsf{if}\ B\ \mathsf{then}\ D\ \mathsf{else}\ E)\ \mathsf{else}\ (\mathsf{if}\ C\ \mathsf{then}\ D\ \mathsf{else}\ E)$

3. $\mathsf{if}\ A\ \mathsf{then}\ (\mathsf{if}\ B\ \mathsf{then}\ D\ \mathsf{else}\ E)\ \mathsf{else}\ (\mathsf{if}\ C\ \mathsf{then}\ D\ \mathsf{else}\ E) \nvdash \mathsf{if}\ (\mathsf{if}\ A\ \mathsf{then}\ B\ \mathsf{else}\ C)\ \mathsf{then}\ D\ \mathsf{else}\ E$.

As a matter of fact, either one of the last two rules renders the connective if-then-else classical. This can be observed by taking in (2) $B = \bot, C = \top, D = \bot, E = \top$. Then the left-hand-side is equivalent with $\neg\neg A$ and the right-hand-side is equivalent with A. In (3), take $B = \bot$, $C = \top$, $D = \top$, $E = A$. Then the left-hand-side is equivalent with $\top$ and the right-hand-side with $\neg A \vee A$. So, the addition of either one of these judgments as a rule renders the system classical.

An important property is that (just as in classical logic), the constructive if-then-else, together with $\top$ and $\bot$ is *functionally complete*: all other connectives can be defined in terms of it. We prove this for $\wedge$, $\vee$, $\to$ and $\neg$. A result from Schroeder-Heister [11] implies that all constructive connectives can be defined in terms of if-then-else.

▶ **Definition 10.** The derivation rules for the standard intuitionistic connectives are the following. These rules are derived from the truth tables and optimized following Lemmas 7 and 8. The rules for $\wedge$ are the intro rule of Example 3 and the elimination rules of Example 6. The rules for $\neg$ are given in Example 3. The rules for $\vee$ and $\to$ and $\top$ and $\bot$ are:

$$\frac{\vdash A \vee B \quad A \vdash D \quad B \vdash D}{\vdash D}\ \vee\text{-el} \qquad \frac{\vdash A}{\vdash A \vee B}\ \vee\text{-in}_1 \qquad \frac{\vdash B}{\vdash A \vee B}\ \vee\text{-in}_2 \qquad \frac{}{\vdash \top}\ \top\text{-in}$$

$$\frac{\vdash A \to B \quad \vdash A}{\vdash B} \to\text{-el} \qquad \frac{\vdash B}{\vdash A \to B} \to\text{-in}_1 \qquad \frac{A \vdash A \to B}{\vdash A \to B} \to\text{-in}_2 \qquad \frac{\vdash \bot}{\vdash D}\ \bot\text{-el}$$

▶ **Example 11.** As our only example for classical logic, we give the classical rules for implication. The elimination rule is the same, $\to$-el above, and we also have the first introduction rule $\to$-in$_1$, but in addition we have the rule $\to$-in$_2^c$. We observe that this rule is classical in the sense that one can derive Peirce's law, without using negation. See the derivation below, of Peirce's law.

$$\frac{\dfrac{A \vdash D \quad A \to B \vdash D}{\vdash D} \to\text{-in}_2^c}{} \qquad \frac{\dfrac{\dfrac{\dfrac{(A \to B) \to A \vdash (A \to B) \to A \quad A \to B \vdash A \to B}{A \to B, (A \to B) \to A \vdash A} \to\text{-el}}{A \to B, (A \to B) \to A \vdash ((A \to B) \to A) \to A} \to\text{-in}_1}{A \to B \vdash ((A \to B) \to A) \to A} \to\text{-in}_2}{}$$

$$\frac{\dfrac{A \vdash A}{A \vdash ((A \to B) \to A) \to A} \qquad A \to B \vdash ((A \to B) \to A) \to A}{\vdash ((A \to B) \to A) \to A} \to\text{-in}_2^c$$

▶ **Definition 12.** We define the usual intuitionistic connectives in terms of if-then-else, $\top$ and $\bot$ as follows.

$$A \mathbin{\dot\vee} B \ :=\ A{\to}A/B \qquad A \mathbin{\dot\wedge} B \ :=\ A{\to}B/A \qquad A \mathbin{\dot\to} B \ :=\ A{\to}B/\top \qquad \dot\neg A \ :=\ A{\to}\bot/\top$$

The following is now a routine check.

▶ **Lemma 13.** *The defined connectives in Definition 12 satisfy the derivation rules for these same connectives as given in Definition 10*

▶ **Corollary 14.** *The intuitionistic connective* if-then-else*, together with $\top$ and $\bot$, is functionally complete.*

Proof. Lemma 10 shows that the well-known intuitionistic connectives can all be defined in terms of if-then-else, $\top$ and $\bot$. In [11], it is shown that all connectives can be defined in terms of $\vee$, $\wedge$, $\to$ and $\neg$. ◀

3 Kripke semantics

We now define a Kripke semantics for the intuitionistic rules and prove that it is complete. We follow standard methods, given e.g. in [12, 13], which we generalize to arbitrary connectives. The concrete model we construct to prove completeness is also reminiscent of the *Lindenbaum's lemma* that is used in [7] to prove completeness for a classical semantics. Formulas are built from atoms using existing or defined connectives of any arity, so for each n-ary connective c, we assume a truth table $t_c : \{0,1\}^n \to \{0,1\}$ and we have inductively defined derivability $\vdash$ as a relation between a sets of formulas and a formula above.

▶ **Definition 15.** We define a *Kripke model* as a triple $(W, \leq, \mathsf{at})$ where W is a set of worlds with a reflexive, transitive relation $\leq$ on it and a function $\mathsf{at} : W \to \wp(\mathsf{At})$ satisfying $w \leq w' \Rightarrow \mathsf{at}(w) \subseteq \mathsf{at}(w')$.

In a Kripke model we want to define the relation $w \Vdash \varphi$ between worlds and formulas (φ *is true in world* w). We do this by defining $[\![\varphi]\!]_w \in \{0,1\}$, with the meaning that $[\![\varphi]\!]_w = 1$ if $w \Vdash \varphi$ and $[\![\varphi]\!]_w = 0$ if $w \nVdash \varphi$.

▶ **Definition 16.** Given a Kripke model $(W, \leq, \mathsf{at})$ we define $[\![\varphi]\!]_w \in \{0,1\}$, by induction on φ as follows.

- (atom) if φ is atomic, $[\![\varphi]\!]_w = 1$ iff $\varphi \in \mathsf{at}(w)$.
- (connective) for $\varphi = c(\varphi_1, \ldots, \varphi_n)$, $[\![\varphi]\!]_w = 1$ iff for each $w' \geq w$, $t_c([\![\varphi_1]\!]_{w'}, \ldots, [\![\varphi_1]\!]_{w'}) = 1$ where t_c is the truth table of c.

We define $\Gamma \models \psi$ (ψ is a consequence of Γ) as: for each Kripke model and each world w, if for each φ in Γ, $[\![\varphi]\!]_w = 1$, then $[\![\psi]\!]_w = 1$.

▶ **Lemma 17** (Soundness). *If* $\Gamma \vdash \psi$, *then* $\Gamma \models \psi$

Proof. By induction on the derivation of $\Gamma \vdash \psi$. ◀

Now we prove completeness: if $\Gamma \models \psi$, then $\Gamma \vdash \psi$. We prove this by constructing a special, universal Kripke model.

▶ **Definition 18.** For ψ a formula and Γ a set of formulas, we say that Γ *is a ψ-maximal set of formulas* if $\Gamma \nvdash \psi$ and for every formula $\varphi \notin \Gamma$ we have: $\Gamma, \varphi \vdash \psi$.

Given a formula ψ and a set of formulas Γ such that $\Gamma \nvdash \psi$, we can extend Γ to a ψ-maximal set Γ' that contains Γ as follows. Take an enumeration of the formulas as $\varphi_1, \varphi_2, \ldots$ and define recursively $\Gamma_0 := \Gamma$ and $\Gamma_{i+1} := \Gamma_i$ if $\Gamma_i, \varphi_{i+1} \vdash \psi$ and $\Gamma_{i+1} := \Gamma_i, \varphi_{i+1}$ if $\Gamma_i, \varphi_{i+1} \nvdash \psi$. Then take $\Gamma' := \bigcup_{i \in \mathbf{N}} \Gamma_i$. (NB. as always, Γ_i, φ_{i+1} denotes $\Gamma_i \cup \{\varphi_{i+1}\}$.)

▶ Fact 19. We list a couple of simple important facts about ψ-maximal sets Γ.

1. For every φ, we have $\varphi \in \Gamma$ or $\Gamma, \varphi \vdash \psi$.
2. So, for every φ, if $\varphi \notin \Gamma$ then $\Gamma, \varphi \vdash \psi$.
3. For every φ, if $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$. (This follows by Lemma 4, taking $\Delta = \Gamma$. If $\varphi \notin \Gamma$, then $\Gamma, \varphi \vdash \psi$ which together with $\Gamma \vdash \varphi$ yields $\Gamma \vdash \psi$).

▶ **Definition 20.** We define the Kripke model $U = (W, \leq, \mathsf{at})$ as follows:

- A world $w \in W$ is a pair (Γ, ψ) where Γ is a ψ-maximal set of formulas.
- $(\Gamma, \psi) \leq (\Gamma', \psi') := \Gamma \subseteq \Gamma'$.
- $\mathsf{at}(\Gamma, \psi) := \Gamma \cap \mathsf{At}$.

▶ **Lemma 21.** *In the model U we have, for all worlds $(\Gamma, \psi) \in W$:*

$$\forall \varphi, \varphi \in \Gamma \Leftrightarrow [\![\varphi]\!]_{(\Gamma,\psi)} = 1.$$

Proof. The proof is by induction on φ. If $\varphi \in \mathsf{At}$, the result is immediate, so suppose that $\varphi = c(\varphi_1, \ldots, \varphi_n)$ where c has truth table t_c. We prove the two directions separately.

($\Rightarrow$): Assume $\varphi \in \Gamma$.

We have $[\![\varphi]\!]_{(\Gamma,\psi)} = 1$ iff for all $\Gamma' \supseteq \Gamma$ and for all ψ', writing $w' = (\Gamma', \psi')$, we have $t_c([\![\varphi_1]\!]_{w'}, \ldots, [\![\varphi_n]\!]_{w'}) = 1$.

So let $\Gamma' \supseteq \Gamma$ and let ψ' be a formula such that Γ' is ψ'-maximal. For the sub-formulas of φ we have the following possibilities

- $[\![\varphi_j]\!]_{w'} = 1$, and then by induction hypothesis: $\varphi_j \in \Gamma'$ and so $\Gamma' \vdash \varphi_j$.
- $[\![\varphi_i]\!]_{w'} = 0$, and then by induction hypothesis: $\varphi_i \notin \Gamma'$ and so $\Gamma', \varphi_i \vdash \psi'$.

This corresponds to an entry in the truth table t_c for the connective c.

Suppose $t_c([\![\varphi_1]\!]_{w'}, \ldots, [\![\varphi_n]\!]_{w'}) = 0$. Then this row in the truth table yields an elimination rule that allows us to prove ψ':

$$\frac{\Gamma' \vdash \varphi \quad \ldots \Gamma' \vdash \varphi_j \text{ (for } \varphi_j \text{ with } [\![\varphi_j]\!]_{w'} = 1) \ldots \quad \ldots \Gamma', \varphi_i \vdash \psi' \text{ (for } \varphi_i \text{ with } [\![\varphi_i]\!]_{w'} = 0) \ldots}{\Gamma' \vdash \psi'}\ \mathrm{el}$$

Note that all hypotheses of the rule are derivable, because $\varphi \in \Gamma'$ and the other hypotheses are derivable by induction. So we have $\Gamma' \vdash \psi'$. Contradiction! So: $t_c([\![\varphi_1]\!]_{w'}, \ldots, [\![\varphi_n]\!]_{w'}) = 1$, what we needed to prove.

($\Leftarrow$): Assume $[\![\varphi]\!]_{(\Gamma,\psi)} = 1$ and suppose (towards a contradiction) $\varphi \notin \Gamma$.

Then $\Gamma \nvdash \varphi$ (because if $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$ by the facts we remarked about Kripke model U.) So there is a φ-maximal theory $\Gamma' \supseteq \Gamma$ with $\Gamma' \nvdash \varphi$. So $(\Gamma,' \varphi)$ is a world in U with $(\Gamma, \psi) \leq (\Gamma', \varphi)$. We write $w' := (\Gamma', \varphi)$ and we have

$$t_c([\![\varphi_1]\!]_{w'}, \ldots, [\![\varphi_n]\!]_{w'}) = 1.$$

We consider the different sub-formulas of φ:

- the φ_j with $[\![\varphi_j]\!]_{w'} = 1$, and so (by induction hypothesis) $\varphi_j \in \Gamma'$ and so $\Gamma' \vdash \varphi_j$;
- the φ_i with $[\![\varphi_i]\!]_{w'} = 0$, and so (by induction hypothesis) $\varphi_i \notin \Gamma'$ and so $\Gamma', \varphi_i \vdash \varphi$.

Now, using an introduction rule for connective c, we can derive φ:

$$\frac{\Gamma' \vdash \varphi_j \text{ (for } \varphi_j \text{ with } [\![\varphi_j]\!]_{w'} = 1) \ldots \quad \ldots \Gamma', \varphi_i \vdash \varphi \text{ (for } \varphi_i \text{ with } [\![\varphi_i]\!]_{w'} = 0) \ldots}{\Gamma' \vdash \varphi}\ \mathrm{in}$$

So we have $\Gamma' \vdash \varphi$, because the hypotheses of the rule are all derivable. Contradiction! So $\varphi \in \Gamma'$. ◀

▶ **Theorem 22.** *If* $\Gamma \models \psi$*, then* $\Gamma \vdash \psi$.

Proof. Suppose $\Gamma \models \psi$ and $\Gamma \nvdash \psi$. Then we can find a ψ-maximal superset Γ' of Γ such that $\Gamma' \nvdash \psi$. In particular: ψ is not in Γ'. So (Γ', ψ) is a world in the Kripke model U in which each member of Γ is true: $[\![\varphi]\!]_{(\Gamma',\psi)} = 1$ for all $\varphi \in \Gamma$, by Lemma 21. However, ψ is not true in (Γ', ψ): $[\![\psi]\!]_{(\Gamma',\psi)} = 0$. So $\Gamma \not\models \psi$. Contradiction, so $\Gamma \vdash \psi$. ◀

In intuitionistic logic, the disjunction connective has a special property that does not hold for classical logic, called the *disjunction property*: If $\vdash A \vee B$, then $\vdash A$ or $\vdash B$. This implies that the disjunction is "strong": if one has a proof of a disjunction, one has a proof of one of the disjoints. (Which is classically not the case, viz. $\vdash A \vee \neg A$.) The disjunction property can easily be proved using Kripke semantics, relying on the completeness theorem. We want to generalize this to our new connectives and we introduce the notion of a *splitting connective*.

▶ **Definition 23.** Let c be an n-ary connective, $1 \leq i, j \leq n$. We say that c *is* i, j*-splitting* in case the truth table for c has the following shape

p_1	$\dots$	p_i	$\dots$	p_j	$\dots$	p_n	$c(p_1,\dots,p_n)$
$-$	$\dots$	0	$\dots$	0	$\dots$	$-$	0
$-$	$\dots$	0	$\dots$	0	$\dots$	$-$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-$	$\dots$	0	$\dots$	0	$\dots$	$-$	0
$-$	$\dots$	0	$\dots$	0	$\dots$	$-$	0

So, in all rows where $p_i = p_j = 0$ we have $c(p_1, \dots, p_n) = 0$. Phrased purely in terms of t_c, that is:

$$t_c(p_1, \dots, p_{i-1}, 0, p_{i+1}, \dots, p_{j-1}, 0, p_{j+1}, \dots, p_n) = 0$$

for all $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{j-1}, p_{j+1}, \dots, p_n \in \{0, 1\}$.

Note that a connective can be i, j-splitting for more than one i, j-pair. Examples are the ternary connectives most and if-then-else. See Appendix 8.

In [12] and [13], the completeness of Kripke semantics is proved using *prime theories* (which in [12] are called *saturated theories*.) A theory is a set of formulas Γ that is closed under $\vdash$ and a prime theory is defined as a theory that satisfies the *disjunction property*: if $\Gamma \vdash A \vee B$, then $\Gamma \vdash A$ or $\Gamma \vdash B$. (This is equivalent to $A \vee B \in \Gamma$ implies $A \in \Gamma$ or $B \in \Gamma$.) We generalize the disjunction property to arbitrary n-ary intuitionistic connectives.

▶ **Definition 24.** A *theory* is a set of formulas Γ that is closed under $\vdash$. We say that Γ is a *prime theory* if for all i, j-splitting connectives c, in case $c(A_1, \dots, A_n) \in \Gamma$, then $A_i \in \Gamma$ or $A_j \in \Gamma$.

▶ **Lemma 25.** *If Γ is ψ-maximal then Γ is a prime theory.*

Proof. Let Γ be a ψ-maximal set of formulas. Obviously, Γ is closed under derivability, so it is a theory. Let c be an i, j-splitting connective and let $\varphi = c(A_1, \dots, A_n) \in \Gamma$. Suppose $A_i \notin \Gamma$ and $A_j \notin \Gamma$. Because Γ is ψ-maximal, this means that $\Gamma, A_i \vdash \psi$ and $\Gamma, A_j \vdash \psi$. For the other formulas $A_k \in \{A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_j, A_{j+1}, \dots, A_n\}$ we do not know whether $A_k \in \Gamma$ (and then $\Gamma \vdash A_k$) or $A_k \notin \Gamma$ (and then $\Gamma, A_k \vdash \psi$), but for each A_k either one of the two is the case. Because c is i, j-splitting, we have an elimination rule

$$\frac{\Gamma \vdash \varphi \quad \dots \quad \Gamma, A_i \vdash \varphi \quad \dots \quad \Gamma, A_j \vdash \varphi \quad \dots}{\Gamma \vdash \psi}\ \text{el}$$

All hypotheses are derivable, so the conclusion is derivable. Contradiction! So $A_i \in \Gamma$ or $A_j \in \Gamma$. ◀

We now prove our generalization of the disjunction property.

▶ **Lemma 26.** *Let c be an i, j-splitting connective and suppose $\vdash c(A_1, \dots, A_n)$. Then $\vdash A_i$ or $\vdash A_j$.*

Proof. Let c be an i, j-splitting connective and let $\varphi = c(A_1, \dots, A_n)$ be a formula with $\vdash \varphi$.

Suppose $\nvdash A_i$ and $\nvdash A_j$. Then there are Kripke models K_1 and K_2 such that $K_1 \nVdash A_i$ and $K_2 \nVdash A_j$. We may assume that the sets of worlds of K_1 and K_2 are disjoint so we can construct a Kripke model K as the union of K_1 and K_2 where we add a special "root world" w_0 that is below all worlds of K_1 and K_2, with $\mathsf{at}(w_0) = \emptyset$. It is easily verified that K is a Kripke model and we have $w_0 \nVdash A_i$, because w_0 is below some world w in K_1 with $w \nVdash A_i$; similarly $w_0 \nVdash A_j$. So, $[\![A_i]\!]_{w_0} = [\![A_j]\!]_{w_0} = 0$. But then $w_0 \nVdash \varphi$, because

$[\![\varphi]\!]_{w_0} = [\![c(A_1,\ldots,A_n)]\!]_{w_0} = 1$ iff for all $w \geq w_0$: $t_c([\![A_1]\!]_w,\ldots,[\![A_n]\!]_w) = 1$. However, for $w := w_0$, whatever the values of $[\![A_k]\!]_w$ are for $k \neq i,j$, $t_c([\![A_1]\!]_w,\ldots,[\![A_n]\!]_w) = 0$. On the other hand, $w_0 \Vdash \varphi$, because $\vdash \varphi$, so we have a contradiction. We conclude that $\vdash A_i$ or $\vdash A_j$.

◀

4 Cuts and cut-elimination

The idea of a cut in intuitionistic logic is an introduction of a formula Φ immediately followed by an elimination of Φ. We will call this a *direct intuitionistic cut*. In general in between the intro rule for Φ and the elim rule for Φ, there may be other auxiliary rules, so occasionally we may have to first permute the elim rule with these auxiliary rules to obtain a direct cut that can be contracted. We leave that for future research and now just define the notion of direct cut and contraction of a direct cut.

▶ **Definition 27.** Let c be a connective of arity n, with an elim rule and an intuitionistic intro rule derived from the truth table, as in Definition 1. So suppose we have the following rules in the truth table t_c.

p_1	$\ldots$	p_n	$c(p_1,\ldots,p_n)$
a_1	$\ldots$	a_n	0
b_1	$\ldots$	b_n	1

An *intuitionistic direct cut* in a derivation is a pattern of the following form, where $\Phi = c(A_1,\ldots,A_n)$.

$$\dfrac{\dfrac{\ldots\ \begin{matrix}\vdots\,\Sigma_j\\ \Gamma\vdash A_j\end{matrix}\ \ldots\quad \ldots\ \begin{matrix}\vdots\,\Sigma_i\\ \Gamma, A_i\vdash \Phi\end{matrix}\ \ldots}{\Gamma\vdash\Phi}\qquad \ldots\ \begin{matrix}\vdots\,\Pi_k\\ \Gamma\vdash A_k\end{matrix}\ \ldots\quad \ldots\ \begin{matrix}\vdots\,\Pi_\ell\\ \Gamma, A_\ell\vdash D\end{matrix}\ \ldots}{\Gamma\vdash D}$$

- A_j ranges over all propositions where $b_j = 1$; A_i ranges over all propositions where $b_i = 0$,
- A_k ranges over all propositions where $a_k = 1$; A_ℓ over all propositions where $a_\ell = 0$,

The *elimination of a direct cut* is defined by replacing the derivation pattern above by

1. If $\ell = j$ (for some ℓ, j):

$$\begin{matrix}\begin{matrix}\vdots\,\Sigma_j\\ \Gamma\vdash A_j\end{matrix}\ \ldots\ \begin{matrix}\vdots\,\Sigma_j\\ \Gamma\vdash A_j\end{matrix}\\ \vdots\,\Pi_\ell\\ \Gamma\vdash D\end{matrix}$$

2. If $k = i$ (for some k, i):

$$\dfrac{\begin{matrix}\begin{matrix}\vdots\,\Pi_k\\ \Gamma\vdash A_i\end{matrix}\ \ldots\ \begin{matrix}\vdots\,\Pi_k\\ \Gamma\vdash A_i\end{matrix}\\ \vdots\,\Sigma_i\\ \Gamma\vdash\Phi\end{matrix}\qquad \ldots\ \begin{matrix}\vdots\,\Pi_k\\ \Gamma\vdash A_i\end{matrix}\ \ldots\quad \ldots\ \begin{matrix}\vdots\,\Pi_\ell\\ \Gamma, A_\ell\vdash D\end{matrix}\ \ldots}{\Gamma\vdash D}$$

There may be several choices for the i and j in the previous definition, so cut-elimination is non-deterministic in general. We study the example of if-then-else in more detail.

▶ **Example 28.** The intuitionistic cut-elimination rules for if-then-else are the following.

(then-then)

$$\dfrac{\dfrac{\Gamma\vdash A \qquad \begin{matrix}\vdots\,\Sigma\\ \Gamma\vdash B\end{matrix}}{\Gamma\vdash A\to B/C} \qquad \Gamma\vdash A}{\Gamma\vdash B} \qquad\mapsto\qquad \begin{matrix}\vdots\,\Sigma\\ \Gamma\vdash B\end{matrix}$$

(else-then)

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\Sigma\\ \Gamma, A\vdash A\to B/C\end{matrix} \qquad \Gamma\vdash C}{\Gamma\vdash A\to B/C} \qquad \begin{matrix}\vdots\,\Pi\\ \Gamma\vdash A\end{matrix}}{\Gamma\vdash B} \qquad\mapsto\qquad \dfrac{\begin{matrix}\vdots\,\Pi & & \vdots\,\Pi\\ \Gamma\vdash A & \dots & \Gamma\vdash A\\ & \vdots\,\Sigma & \\ & \Gamma\vdash A\to B/C & \end{matrix} \qquad \begin{matrix}\vdots\,\Pi\\ \Gamma\vdash A\end{matrix}}{\Gamma\vdash B}$$

(then-else)

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\Sigma\\ \Gamma\vdash A\end{matrix} \qquad \Gamma\vdash B}{\Gamma\vdash A\to B/C} \qquad \begin{matrix}\vdots\,\Pi\\ \Gamma, A\vdash D\end{matrix} \qquad \Gamma, C\vdash D}{\Gamma\vdash D} \qquad\mapsto\qquad \begin{matrix}\vdots\,\Sigma & & \vdots\,\Sigma\\ \Gamma\vdash A & \dots & \Gamma\vdash A\\ & \vdots\,\Pi & \\ & \Gamma\vdash D & \end{matrix}$$

(else-else)

$$\dfrac{\dfrac{\Gamma, A\vdash A\to B/C \qquad \begin{matrix}\vdots\,\Sigma\\ \Gamma\vdash C\end{matrix}}{\Gamma\vdash A\to B/C} \qquad \Gamma, A\vdash D \qquad \begin{matrix}\vdots\,\Pi\\ \Gamma, C\vdash D\end{matrix}}{\Gamma\vdash D} \qquad\mapsto\qquad \begin{matrix}\vdots\,\Sigma & & \vdots\,\Sigma\\ \Gamma\vdash C & \dots & \Gamma\vdash C\\ & \vdots\,\Pi & \\ & \Gamma\vdash D & \end{matrix}$$

5 A Curry-Howard isomorphism

We now define types terms for derivations, which enables the study of "proofs as terms" and emphasizes the computational interpretation of proofs. Here, we only define terms for derivations in the inituitionistic logic, which can be extended to the classical logic in an obvious way. We first define terms associated with connectives in general. Then, to show the usefulness of our approach to logical connectives, we will focus on the if-then-else connective.

▶ **Definition 29.** Suppose we have a logic with intuitionistic derivation rules, as derived from truth tables for a set of connectives $\mathcal{C}$, as in Definition 1. We define rules for the judgment $\Gamma \vdash t : A$, where A is a formula, Γ is a set of declarations $\{x_1 : A_1, \dots, x_m : A_m\}$, where the A_i are formulas and the x_i are term-variables such that every x_i occurs at most once in Γ, and t is a *proof-term* as follows. For every connective $c \in \mathcal{C}$ of arity n, we have an *introduction term* $\iota(t_1, \dots, t_n)$ and an *elimination term* $\varepsilon(t_0, t_1, \dots, t_n)$, where the t_i are again terms of the shape $\lambda x.t'$, where x is a term-variable and t' is a term. The terms are *typed* using the following derivation rules.

$$\dfrac{}{\Gamma\vdash x_i : A_i}\; x_i : A_i \in \Gamma$$

$$\dfrac{\dots \Gamma\vdash t_j : A_j \dots \qquad \dots \Gamma, y_i : A_i \vdash q_i : \Phi \dots}{\Gamma\vdash \iota(\overrightarrow{t}, \overrightarrow{\lambda y.q}) : \Phi}\;\text{in}$$

$$\frac{\Gamma \vdash t_0 : \Phi \ldots \quad \ldots \Gamma \vdash t_k : A_k \ldots \quad \ldots \Gamma, y_\ell : A_\ell \vdash q_\ell : D}{\Gamma \vdash \varepsilon(t_0, \overrightarrow{t}, \overrightarrow{\lambda y.q}) : D}\text{ el}$$

Here, $\overrightarrow{t}$ is the sequence of terms $t_1, \ldots, t_p$ for all the 1-entries in the truth table, and $\overrightarrow{\lambda y.q}$ is the sequence of terms $\lambda y_1.q_1, \ldots, \lambda y_r.q_r$ for all the 0-entries in the truth table.

One may think of the λ-abstracted variables as being *typed* so then one could write $\overrightarrow{\lambda y : A.q}$ and $\lambda y_1 : A_1.q_1, \ldots, \lambda y_r : A_r.q_r$. However, this clutters up the syntax considerably, so we will leave these types implicit. Moreover, decidability of typing, or a typing algorithm for (untyped) terms of the calculus is not our concern here.

There are term reduction rules that correspond to the elimination of direct cuts.

▶ **Definition 30.** Given a direct cut as defined in Definition 27, we add reduction rules for the associated terms as follows. (For simplicity of presentation we write the "matching cases" in Definition 27 as last term of the sequence.)

$$\begin{array}{lcl} & \text{For the } \ell = j \text{ case:} & \\ \varepsilon(\iota(\overrightarrow{t}, t_j, \overrightarrow{\lambda y.q}), \overrightarrow{s}, \overrightarrow{\lambda y.r}, \lambda y_\ell.r_\ell) & \longrightarrow & r_\ell[y_\ell := t_j] \\ & \text{For the } k = i \text{ case:} & \\ \varepsilon(\iota(\overrightarrow{t}, \overrightarrow{\lambda y.q}, \lambda y_i.q_i), \overrightarrow{s}, s_k, \overrightarrow{\lambda y.r}) & \longrightarrow & \varepsilon(q_i[y_i := s_k], \overrightarrow{s}, s_k, \overrightarrow{\lambda y.r}) \end{array}$$

The reduction is extended in the straightforward way to sub-terms, by defining it as a congruence with respect to the term constructions.

This Definition gives a reduction rule for every combination of an elimination and an introduction. For an n-ary connective, there are 2^n rules in the truth table, and therefore 2^n constructors (introduction plus elimination constructors). Often, we will want to just look at the optimized rules, following Lemmas 7 and 8. For these optimized rules, there is also a straightforward definition of proof-terms and of the reduction relation associated with cut-elimination. The Lemmas 7 and 8 can be extended to terms and reductions: the proof-terms for the optimized rules can be defined in terms of the terms for the original calculus, and the reduction rules for the optimized proof terms are an instance of reductions in the original calculus (often multi-step).

We now focus on the logic with just if-then-else. We define a calculus λif-then-else for proof terms for this logic.

▶ **Definition 31.** We define the calculus λif-then-else as a calculus for terms and reductions for the if-then-else logic as follows. (To understand the reduction rules, also look at Section 2.1.)

$$\frac{\Gamma \vdash t_0 : A{\to}B/C \quad \Gamma \vdash a : A}{\Gamma \vdash \varepsilon_1(t_0, a) : B}\text{ then-el} \qquad \frac{\Gamma \vdash t_0 : A{\to}B/C \quad \Gamma, x : A \vdash t : D \quad \Gamma, y : C \vdash q : D}{\Gamma \vdash \varepsilon_2(t_0, \lambda x.t, \lambda y.q) : D}\text{ else-el}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash \iota_1(a, b) : A{\to}B/C}\text{ then-in} \qquad \frac{\Gamma, x : A \vdash t : A{\to}B/C \quad \Gamma \vdash c : C}{\Gamma \vdash \iota_2(\lambda x.t, c) : A{\to}B/C}\text{ else-in}$$

The reduction rules are

$$\begin{array}{lcl@{\qquad}lcl} \varepsilon_1(\iota_1(a, b), a') & \longrightarrow & b & \varepsilon_1(\iota_2(\lambda x.t, c), a) & \longrightarrow & \varepsilon_1(t[x := a], a) \\ \varepsilon_2(\iota_1(a, b), \lambda x.t, \lambda y.q) & \longrightarrow & t[x := a] & \varepsilon_2(\iota_2(\lambda x.t, c), \lambda z.d, \lambda y.q) & \longrightarrow & q[y := c] \end{array}$$

Here $t[x := a]$ denotes the substitution of a for x in t.

The interpretation of intuitionistic proposition logic can be extended to include proof terms. These are well-known for proposition logic. e.g. see [13]. The interpretation of Definition 12 can be extended to the proof terms. If we denote this translation by a $(\dot{-})$, we find that, if $\Gamma \vdash t : A$ in intuitionistic proposition logic with proof terms, then $\dot{\Gamma} \vdash \dot{t} : \dot{A}$ in the calculus λif-then-else of Definition 31. Lemma 13 also extend to term reductions: If $t \longrightarrow q$ for the proof terms t and q in intuitionistic proposition logic, then $\dot{t} \longrightarrow^{+} \dot{q}$ in λif-then-else (where $\longrightarrow^{+}$ denotes the transitive closure of $\longrightarrow$).

We do not define the $(\dot{-})$ function precisely, nor do we prove the mentioned properties about it here. The reason is that we merely want to use it as one of the motivations for studying *strong normalization* of reduction of proof terms of λif-then-else. Strong normalization is the property that a term has no infinite reduction starting from it. Strong normalization for λif-then-else implies (using the properties of the $(\dot{-})$ function) strong normalization of proof term reduction in intuitionistic proposition logic.

5.1 Strong Normalization

We prove SN by adapting the well-known *saturated sets method* of Tait to our calculus. We write SN for the set of strongly normalizing (untyped) terms and we write Term for the set of all untyped terms and Var for the set of variables.

▶ **Definition 32.**
1. The set Neut of *neutral terms* is defined by (a) $\mathsf{Var} \subseteq \mathsf{Neut}$, (b) $\varepsilon_1(t_0, a) \in \mathsf{Neut}$ for all $t_0 \in \mathsf{Neut}$ and $a \in \mathsf{SN}$, (c) $\varepsilon_2(t_0, \lambda x.t, \lambda y.q) \in \mathsf{Neut}$ for all $t_0 \in \mathsf{Neut}$, $t, q \in \mathsf{SN}$.
2. The term t does a *key reduction* to q, notation $t \longrightarrow_k q$, in case (i) t is a redex itself (according to Definition 31) and q is its reduct, or (ii) $t = \varepsilon_1(t_0, a)$, $q = \varepsilon_1(q_0, a)$ and $t_0 \longrightarrow_k q_0$, or (iii) $t = \varepsilon_2(t_0, \lambda x.r, \lambda y.s)$, $q = \varepsilon_2(q_0, \lambda x.r, \lambda y.s)$ and $t_0 \longrightarrow_k q_0$.
3. A set $X \subseteq \mathsf{Term}$ is *saturated* if it satisfies the following properties (i) $X \subseteq \mathsf{SN}$, (ii) $\mathsf{Neut} \subseteq X$ and (iii) X is closed under *key-redex expansion*: if $q \in X$, $t \in \mathsf{SN}$ and $t \longrightarrow_k q$, then $t \in X$.
4. Given $X, Y, Z \in \mathsf{SAT}$ we define the set $X {\to} Y/Z$ by

$$\begin{aligned} X{\to}Y/Z := \{M \mid \quad & \forall a \in X(\varepsilon_1(M,a) \in Y) \ \wedge\ \forall D \in \mathsf{SAT}, \forall t, q \in \mathsf{Term}, \\ & \forall a \in X(t[x := a] \in D) \ \wedge\ \forall c \in Z(q[y := c] \in D) \\ & \Longrightarrow \varepsilon_2(M, \lambda x.t, \lambda y.q) \in D\ \} \end{aligned}$$

▶ **Lemma 33.** *If* $X, Y, Z \in \mathsf{SAT}$*, then* $X{\to}Y/Z \in \mathsf{SAT}$*.*

See Appendix 8 for a proof.

We use the saturated sets as a semantics for types: if A is a type, $\langle A \rangle$ will be a saturated set. The simplest way to do this is to interpret all type variables (proposition letters) as the set SN (which is indeed a saturated set) and interpret $A{\to}B/C$ as $\langle A\rangle{\to}\langle B\rangle/\langle C\rangle$, where this definition is from Definition 32.

▶ **Definition 34.** Given a context Γ, a map (valuation) $\rho : \mathsf{Var} \to \mathsf{Term}$ satisfies Γ, notation $\rho \models \Gamma$, in case $\rho(x) \in \langle A \rangle$ for all $x : A \in \Gamma$.

If $t \in \mathsf{Term}$ and $\rho : \mathsf{Var} \to \mathsf{Term}$, we write $\langle t \rangle_\rho$ for t where ρ has been carried out as a substitution on t.

A valuation $\rho : \mathsf{Var} \to \mathsf{Term}$ is only relevant for a finite number of variables: those that are declared in the context Γ under consideration. So we will always assume that $\rho(x) \neq x$ only for a finite number of $x \in \mathsf{Var}$. Those x we call the *support of* ρ. When applying ρ as a substitution to a term t we may need to "go under a λ", e.g. when applying ρ to $\iota_2(\lambda x.t, c)$ In this case we always assume that the bound variable is not in the support of ρ. (We can always rename it.) This allows us to just write $\langle \iota_2(\lambda x.t, c)\rangle_\rho = \iota_2(\lambda x.\langle t\rangle_\rho, \langle c\rangle_\rho)$.

▶ **Lemma 35.** *If* $\Gamma \vdash t : A$*, and* $\rho \models \Gamma$*, then* $\langle t\rangle_\rho \in \langle A\rangle$*.*

For a proof, see Appendix 8

The following is now immediate by taking $\rho(x) := x$ for all $x \in \mathsf{Var}$. Because $\mathsf{Var} \subseteq \mathsf{Neut} \subseteq \langle A\rangle$, we know that $\rho \models \Gamma$. So, if $\Gamma \vdash t : A$, then $\langle t\rangle_\rho = t \in \langle A\rangle \subseteq \mathsf{SN}$.

▶ **Corollary 36.** *The system* λif-then-else *is strongly normalizing: all reductions on proof terms are finite.*

6 Conclusion and Further work

We have introduced a general procedure for deriving natural deduction rules from truth tables that applies both to classical and intuitionistic logic. Our deduction rules obey a specific format, making it easier to study. To show that the intuitionistic rules are truly constructive we have defined a complete Kripke semantics for the intuitionistic rules. We have defined cut-elimination for intuitionistic logic in general and given a Curry-Howard proofs-as-terms isomorphism for it. We have studied it in more detail for if-then-else and proved the reduction on proof-terms to be strongly normalizing.

The work described here raises very many new research questions: Is cut-elimination normalizing in general for an arbitrary set of connectives? How to define cut-elimination for the classical case, and what is its connection with a term calculus, for example calculi for classical logic studied in [1] and [9]? Also [3] defines a whole series of systems and connectives, of which the $-$ of [2] is just one. How do the computation rules compare with ours?

On a more technical note: In λif-then-else, a cut can be "hidden" in case the second or third sub-derivation of an else-elrule ends with an introduction of a formula $D := E{\to}F/G$ which is eliminated after the else-el. The idea is to permute the elimination over the application of the else-elrule. This can be achieved by the following *permuting reduction rules*:

$$\begin{aligned}
\varepsilon_1(\varepsilon_2(t_0, \lambda x.t, \lambda y.q), e) &\longrightarrow \varepsilon_2(t_0, \lambda x.\varepsilon_1(t, e), \lambda y.\varepsilon_1(q, e))\\
\varepsilon_2(\varepsilon_2(t_0, \lambda x.t, \lambda y.q), \lambda v.r, \lambda z.s) &\longrightarrow \varepsilon_2(t_0, \lambda x.\varepsilon_2(t, \lambda v.r, \lambda z.s), \lambda y.\varepsilon_2(q, \lambda v.r, \lambda z.s))
\end{aligned}$$

These reduction rules correspond to obvious transformations of derivations, permuting one elimination over another. The normal forms t for this combined reduction are such that all sub-terms of t have types that are sub-types of the type of t or sub-types of types of free variables in t. We leave it for future research to prove that λif-then-else with these permuting reductions is normalizing. Techniques as in [6], where a similar property is proved for intuitionistic logic with permuting cuts, may be useful.

7 References

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8 Appendix

Truth tables of most and if-then-else

A	B	C	$\mathsf{most}(A,B,C)$	$A{\to}B/C$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	0
1	1	0	1	1
1	1	1	1	1

We see that most is i,j-splitting for every i,j. Indeed, if $\vdash \mathsf{most}(A,B,C)$, we can derive $\vdash A$ or $\vdash B$ but also $\vdash A$ or $\vdash C$ and also $\vdash B$ or $\vdash C$.

The connective if-then-else is not $1,2$-splitting but it is $1,3$-splitting and $2,3$-splitting: if $\vdash A{\to}B/C$, then we have $\vdash A$ or $\vdash C$ and also $\vdash B$ or $\vdash C$.

From the lines in the truth table of $A{\to}B/C$ with a 0 we get the following four elimination rules:

$$\frac{\vdash A{\to}B/C \quad A\vdash D \quad B\vdash D \quad C\vdash D}{\vdash D} \qquad \frac{\vdash A{\to}B/C \quad A\vdash D \quad \vdash B \quad C\vdash D}{\vdash D}$$

$$\frac{\vdash A{\to}B/C \quad \vdash A \quad B\vdash D \quad C\vdash D}{\vdash D} \qquad \frac{\vdash A{\to}B/C \quad \vdash A \quad B\vdash D \quad \vdash C}{\vdash D}$$

Using Lemmas 7 and 8, these can be reduced to the following two. (The two rules on the first line reduce to else-el, the two rules on the second line reduce to then-el.)

$$\frac{\vdash A{\to}B/C \quad A\vdash D \quad C\vdash D}{\vdash D}\ \text{else-el} \qquad \frac{\vdash A{\to}B/C \quad \vdash A}{\vdash B}\ \text{then-el}$$

From the lines in the truth table of $A \to B/C$ with a 1 we get the following four classical introduction rules:

$$\frac{A \to B/C \vdash D \quad A \vdash D \quad B \vdash D \quad \vdash C}{\vdash D} \qquad \frac{A \to B/C \vdash D \quad A \vdash D \quad \vdash B \quad \vdash C}{\vdash D}$$

$$\frac{A \to B/C \vdash D \quad \vdash A \quad \vdash B \quad C \vdash D}{\vdash D} \qquad \frac{A \to B/C \vdash D \quad \vdash A \quad \vdash B \quad \vdash C}{\vdash D}$$

Using Lemmas 7 and 8 can be reduced to the following two. (The two rules on the first line reduce to else-in, the two rules on the second line reduce to then-in.)

$$\frac{A \to B/C \vdash D \quad A \vdash D \quad \vdash C}{\vdash D}\ \text{else-in}^c \qquad \frac{\vdash A \quad \vdash B}{\vdash A \to B/C}\ \text{then-in}$$

These are the classical rules for if-then-else. Only the last rule, else-in^c, is different from the constructive one, as given in Example 2.1.

Truth tables and rules for subtraction and bi-implication

We now treat subtraction of [2] and bi-implication. The classical reading of $A - B$ is $A \wedge \neg B$, so we have the truth table below.

A	B	$A - B$	$A \leftrightarrow B$
0	0	0	1
0	1	0	0
1	0	1	0
1	1	0	1

This yields the following intuitionistic derivation rules. For subtraction, these are the same as [2].

$$\frac{\vdash A \quad B \vdash A - B}{\vdash A - B}\ -\text{in} \qquad \frac{\vdash A - B \quad A \vdash D}{\vdash D}\ -\text{el}_1 \qquad \frac{\vdash A - B \quad \vdash B}{\vdash D}\ -\text{el}_2$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \leftrightarrow B}\ \leftrightarrow \text{in}_1 \qquad \frac{A \vdash A \leftrightarrow B \quad B \vdash A \leftrightarrow B}{\vdash A \leftrightarrow B}\ \leftrightarrow \text{in}_2$$

$$\frac{\vdash A \leftrightarrow B \quad A \vdash D \quad \vdash B}{\vdash D}\ \leftrightarrow \text{el}_1 \qquad \frac{\vdash A \leftrightarrow B \quad \vdash A \quad B \vdash D}{\vdash D}\ \leftrightarrow \text{el}_2$$

Proof of Lemma 33

If $X, Y, Z \in \mathsf{SAT}$, then $X \to Y/Z \in \mathsf{SAT}$.

Proof. We check the 3 conditions for $X \to Y/Z \in \mathsf{SAT}$.

(i) $X \to Y/Z \subseteq \mathsf{SN}$ follows directly from the fact that if $M \in X \to Y/Z$, then $\varepsilon_1(M, x) \in Y$ and $Y \subseteq \mathsf{SN}$, so $\varepsilon_1(M, x) \in \mathsf{SN}$, so $M \in \mathsf{SN}$.

(ii) We check the inductive cases for Neut:

(a) $\mathsf{Var} \subseteq X \to Y/Z$ because for $z \in \mathsf{Var}$ and $a \in X$, $\varepsilon_1(z, a) \in \mathsf{Neut} \subseteq Y$ and for $D \in \mathsf{SAT}$ and $t, q \in \mathsf{SN}$, $\varepsilon_2(z, \lambda x.t, \lambda y.q) \in \mathsf{Neut} \subseteq D$.
(b) if $M = \varepsilon_1(t_0, a)$, then for all $a' \in X$, $\varepsilon_1(\varepsilon_1(t_0, a), a') \in \mathsf{Neut} \subseteq Y$ and for all $D \in \mathsf{SAT}$ and $t, q \in \mathsf{SN}$, we have $\varepsilon_2(\varepsilon_1(t_0, a), \lambda x.t, \lambda y.q) \in \mathsf{Neut} \subseteq D$.
(c) if $M = \varepsilon_2(t_0, \lambda x.t, \lambda y.q)$ then for all $a' \in X$, $\varepsilon_1(\varepsilon_2(t_0, a), \lambda x.t, \lambda y.q), a') \in \mathsf{Neut} \subseteq Y$. For $D \in \mathsf{SAT}$ and $t', q' \in \mathsf{SN}$ we have $\varepsilon_2(\varepsilon_2(t_0, \lambda x.t, \lambda y.q), \lambda x'.t', \lambda y'.q') \in \mathsf{Neut} \subseteq D$.

(iii) Suppose $M' \longrightarrow_k M$ with $M \in X \to Y/Z$. Then for all $a \in X$, $\varepsilon_1(M', a) \longrightarrow_k \varepsilon_1(M, a)$ and $\varepsilon_1(M, a) \in Y$, so $\varepsilon_1(M', a)$. Similarly, M' satisfies the second condition in the definition of $X \to Y/Z$, so $M' \in X \to Y/Z$.

◀

Proof of Lemma 35

If $\Gamma \vdash t : A$, and $\rho \models \Gamma$, then $\langle t \rangle_\rho \in \langle A \rangle$.

Proof. By induction on the derivation of $\Gamma \vdash t : A$. Suppose $\rho \models \Gamma$. For the (axiom) case, it is trivial.

- Suppose

$$\frac{\Gamma \vdash t_0 : A \to B/C \quad \Gamma \vdash a : A}{\Gamma \vdash \varepsilon_1(t_0, a) : B} \text{ then-el}$$

Then $\langle \varepsilon_1(t_0, a) \rangle_\rho = \varepsilon_1(\langle t_0 \rangle_\rho, \langle a \rangle_\rho) \in \langle B \rangle$ by $\langle t_0 \rangle_\rho \in \langle A \to B/C \rangle$ and the definition of $\langle A \to B/C \rangle$.

- Suppose

$$\frac{\Gamma \vdash t_0 : A \to B/C \quad \Gamma, x : A \vdash t : D \quad \Gamma, y : C \vdash q : D}{\Gamma \vdash \varepsilon_2(t_0, \lambda x.t, \lambda y.q) : D} \text{ else-el}$$

Then $\langle \varepsilon_2(t_0, \lambda x.t, \lambda y.q) \rangle_\rho = \varepsilon_2(\langle t_0 \rangle_\rho, \lambda x.\langle t \rangle_\rho, \lambda y.\langle q \rangle_\rho) \in \langle D \rangle$ by $\langle t_0 \rangle_\rho \in \langle A \to B/C \rangle$ and the definition of $\langle A \to B/C \rangle$.

- Suppose

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash \iota_1(a, b) : A \to B/C} \text{ then-in}$$

Let $a' \in \langle A \rangle$. Then $\varepsilon_1(\iota_1(\langle a \rangle, \langle b \rangle_\rho), a') \longrightarrow_k \langle b \rangle_\rho \in B$, so $\varepsilon_1(\iota_1(\langle a \rangle, \langle b \rangle_\rho), a') \in B$, because $\varepsilon_1(\iota_1(\langle a \rangle, \langle b \rangle_\rho), a') \in \mathsf{SN}$.
Let $D \in \mathsf{SAT}$ and let t and q satisfy $\forall a' \in X(t[x := a'] \in D)$ and $\forall c \in Z(q[y := c] \in D)$.
We have $\varepsilon_2(\iota_1(\langle a \rangle_\rho, \langle b \rangle_\rho), \lambda x.t, \lambda y.q) \longrightarrow_k t[x := \langle a \rangle_\rho] \in D$, because $\varepsilon_2(\iota_1(\langle a \rangle_\rho, \langle b \rangle_\rho), \lambda x.t, \lambda y.q) \in \mathsf{SN}$.
So $\langle \iota_1(a, b) \rangle_\rho = \iota_1(\langle a \rangle_\rho, \langle b \rangle_\rho) \in \langle A \to B/C \rangle$.

- Suppose

$$\frac{\Gamma, z : A \vdash s : A \to B/C \quad \Gamma \vdash c : C}{\Gamma \vdash \iota_1(\lambda z.s, c) : A \to B/C} \text{ else-in}$$

Let $a' \in \langle A \rangle$. Then $\varepsilon_1(\iota_1(\lambda z.\langle s \rangle_\rho, \langle c \rangle_\rho), a') \longrightarrow_k \varepsilon_1(\langle s \rangle_\rho[z := a'], a')$. The induction hypothesis says that $\langle s \rangle_\rho[z := a'] = \langle s \rangle_{\rho[z := a']} \in \langle A \to B/C \rangle$, so $\varepsilon_1(\langle s \rangle_\rho[z := a'], a') \in \langle B \rangle$ and so $\varepsilon_1(\iota_1(\lambda z.\langle s \rangle_\rho, \langle c \rangle_\rho), a') \in \langle B \rangle$, because $\varepsilon_1(\iota_1(\lambda z.\langle s \rangle_\rho, \langle c \rangle_\rho), a') \in \mathsf{SN}$
Let $D \in \mathsf{SAT}$ and let t and q satisfy $\forall a' \in X(t[x := a'] \in D)$ and $\forall c \in Z(q[y := c] \in D)$.
We have $\varepsilon_2(\iota_1(\lambda z.\langle s \rangle_\rho, \langle c \rangle_\rho), \lambda x.t, \lambda y.q) \longrightarrow_k q[y := \langle c \rangle_\rho] \in D$, so
$\varepsilon_2(\iota_1(\lambda z.\langle s \rangle_\rho, \langle c \rangle_\rho), \lambda x.t, \lambda y.q) \in D$, because $\varepsilon_2(\iota_1(\lambda z.\langle s \rangle_\rho, \langle c \rangle_\rho), \lambda x.t, \lambda y.q) \in \mathsf{SN}$.

◀