The logic and mathematics of occasion sentences*

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Abstract

The prime purpose of this paper is, first, to restore to discourse-bound occasion sentences their rightful central place in semantics and secondly, taking these as the basic propositional elements in the logical analysis of language, to contribute to the development of an adequate logic of occasion sentences and a mathematical (Boolean) foundation for such a logic, thus preparing the ground for more adequate semantic, logical and mathematical foundations of the study of natural language. Some of the insights elaborated in this paper have appeared in the literature over the past thirty years, and a number of new developments have resulted from them. The present paper aims at providing an integrated conceptual basis for this new development in semantics. In Section 1 it is argued that the reduction by translation of occasion sentences to eternal sentences, as proposed by Russell and Quine, is semantically and thus logically inadequate. Natural language is a system of occasion sentences, eternal sentences being merely boundary cases. The logic has fewer tasks than is standardly assumed, as it excludes semantic calculi, which depend crucially on information supplied by cognition and context and thus belong to cognitive psychology rather than to logic. For sentences to express a proposition and thus be interpretable and informative, they must first be properly *anchored in context*. A proposition has a truth value when it is, moreover, properly *keyed in the world*, i.e. is about a situation in the world. Section 2 deals with the logical properties of natural language. It argues that presuppositional phenomena require trivalence and presents the trivalent logic PPC_3, with two kinds of falsity and two negations. It introduces the notion of Σ-space for a sentence A (or $/A/$, the set of situations in which A is true) as the basis of logical model theory, and the notion of $/\mathbf{P}^A/$ (the Σ-space of the presuppositions of A), functioning as a 'private' subuniverse for $/A/$. The trivalent Kleene calculus is reinterpreted as a logical account of vagueness, rather than of presupposition. PPC_3 and the Kleene calculus are refinements of standard bivalent logic and can be combined into one logical system. In Section 3 the adequacy of PPC_3 as a truth-functional model of presupposition is considered more closely and given a Boolean foundation. In a noncompositional extended Boolean algebra, three operators are defined: 1_a for the conjoined presuppositions of a, $\widetilde{a}$ for the complement of a within 1_a, and $\widehat{a}$ for the complement of 1_a within Boolean 1. The logical properties of this extended Boolean algebra are axiomatically defined and proved for all possible models. Proofs are provided of the consistency and the completeness of the system. Section 4 is a provisional exploration of the possibility of using the results obtained for a new discourse-dependent account of the logic of modalities in natural language. The overall result is a modified and refined logical and model-theoretic machinery, which takes into account both the discourse-dependency of natural language sentences and the necessity of selecting a key in the world before a truth value can be assigned.

*Although the three authors have worked very much in concert, there has been a clear division of labour. The principal author of the sections 1 and 2 is Seuren, who is also responsible for the notion of 1_a as a noncompositional unary operator on a and accordingly differentiated complement functions. Capretta and Geuvers cast these ideas in a proper mathematical format, with definitions, axioms and proofs in section 3. Seuren's suggestions for section 4 were cast in a preparatory mathematical format by Capretta and Geuvers. All three authors are indebted to Henk Barendregt (Department of Computer Science, Nijmegen University) for his overall support and critical comments.

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Contents

1 Truth and falsity for occasion sentences

In the classical view, which has been accepted since Aristotle's day, truth consists in saying or thinking of what is so, that it is so, and falsity in saying or thinking of what is not so that it is. Truth and falsity are the truth values (TVs), and the bearers of these values, the objects that have the property of being true or false (whether they are objects of speech or of thought) are called *propositions*. This is known as the correspondence theory of truth, and there is little one can say against it, except that it does not give enough.

First of all, if truth is taken to result from correspondence between what is said or thought and what is the case, it is necessary to specify what the correspondence consists in. In other words, an analysis must be provided of what is the case on the one hand, and also of what is said or thought on the other, and elements of the one analysis must then be mapped onto elements of the other, in order to define in precise terms under what conditions there is correspondence. The task of defining such a mapping procedure has occupied many generations of philosophers, but it was not until the 20th century that it was undertaken in a formally precise way, under the name of model-theory, in the context of mathematical logic.

Then there is the built-in ambiguity between saying and thinking: are true or false propositions the result of speech acts or of thought processes? As is argued in Stegmüller (1957, pp. 16–17) and Seuren (1998*b*, pp. 12–18), the correct answer is that a proposition, as a bearer of a TV, is not a linguistic expression, but the result of a mental act of assigning a property to an entity or n-tuple of entities (where both the property and the entities in question may be determinable through a complex process of interpretation). The main argument for this position is the well-known fact that linguistic utterances, in principle, heavily underdetermine their truth conditions, and that the missing elements are supplied by available world and/or situational knowledge. This applies in particular to predicates, whose satisfaction conditions often involve world knowledge. For example, the satisfaction conditions of the predicate *flat* are different in *The front tire was flat* and *The road surface was flat.* Or, to vary on Ryle (1949, p. 24), the prepositional predicate *in* is satisfied under quite different conditions in, for example, *She went out in a red hat* and *She went out in a sports car.*

Truth thus seems to be primarily a cognitive, and not a verbal, notion. This point is important because logic has always, mainly due to the obvious difficulty of analysing thoughts as against the relative accessibility of linguistic structures, operated with a verbal notion of truth, and we shall see presently that this imposes certain limitations.

A third problem lies in the fact that many sentences of natural languages, if taken by themselves and out of context, cannot be assigned a TV. A sentence like:

(1) The girl was right after all.

is a good grammatical sentence of English, with proper English lexical forms, with a subject term and a finite verb form in proper agreement with the subject term, in the simple past tense and with an adjunct of time. But it makes no sense to ask whether it is true or false, until it is known what person is referred to by the subject term, when the event is said to have taken place, and what the issue was that the girl is said to have been right about. We say that this sentence needs a *key* in the real world before it can be assigned a TV.

Sentences that need a key are called *occasion sentences*, whereas sentences that don't are called *eternal sentences* (Quine 1960). Eternal sentences are, in principle, presented in a generic (present) tense and contain no definite but only quantified terms. Thus, a sentence like:

(2) All humans are mortal.

is an eternal sentence and, consequently, it makes perfect sense to ask whether it is true or false, regardless of any context. No specific key is needed in such cases.

Both Aristotelian and modern logic are based exclusively on eternal sentences, the reason being that occasion sentences turn out to pose a number of apparently intractable problems for a sound logic, problems which do not turn up with eternal sentences. Aristotle decided (Metaph $1027^{\text{a-b}}$)

to ban all occasion sentences from his metaphysics and his logic, probably because of the baffling complications which he saw coming with regard to occasion sentences. There is an alternative logical tradition, running from the Stoa through the Middle Ages to the late 19th century, where attempts are made to take occasion sentences into account as well, but this tradition has dried up since 1900, mainly because it was shown in Russell and Whitehead's *Principia Mathematica* that the new Predicate Calculus, restricted as it is to eternal sentences with its quantifiers, variables and logical connectives, is sufficient to express any mathematical proposition. From then on, attempts to account for occasion sentences were given up and logic was exclusively about eternal sentences.

1.1 The translation method is inadequate as a solution for occasion sentences

While the fact that all mathematical propositions can be expressed in terms of eternal sentences in the Russellian Language of Predicate Calculus (LPC) is no doubt of extreme importance, the question of how to determine truth and falsity for occasion sentences, as well as that of their logical properties, remains. The answer provided by modern logic is, in principle, that all occasion sentences must be 'translated' into eternal sentences for which such problems do not exist. This is the basis of the programme initiated by Russell and continued by Quine, who dubbed it the programme of 'elimination of particulars' (Quine 1960). This programme, which underlies virtually all the work done in present-day model-theoretic or 'formal' semantics, is based on two (usually implicit) assumptions. The first is that the 'translations' provided are semantically equivalent to the sentences that have been translated, and the second implies that the logical translations provided will be powerful enough to express any proposition a speaker wishes to express when using a natural language.

These two assumptions have not remained unchallenged. One important problem, directly relevant to the second assumption, but indirectly also to the first, is posed by the so-called 'donkey sentences', so called because of a number of example sentences presented by the British philosopher Walter Burley (±1275–after 1344) in the context of his theory of reference, all containing mention of a donkey. Among Burley's examples is the following (Burley 1988, p. 92):

(3) Omnis homo habens asinum videt illum. (every man who has a donkey sees it)

Burley's problem was that a sentence like (3) will still be true if some man has two donkeys, one that he sees and one that he does not see, as long as every donkey owner has at least one donkey he does see. This would mean that a sentence like 'Some man who has a donkey does not see it' would be compatible with (3) and not be its contradictory. In modern times, the problem was brought up by Geach (1962, pp. 116ff), who re-used Burley's examples (speaking of 'another sort of medieval example', but without mentioning Burley). Geach's donkey-examples were in turn picked up by modern formal semanticists, who found that sentences of the types:

(4) a. If George owns a donkey he feeds *it*.
b. Every farmer who owns a donkey feeds *it*.
c. Either George does not own a donkey or he feeds *it*.

cannot be translated into LPC, which allows for only two kinds of terms, (bound) variables and constant terms that refer to a reference object. The pronoun *it* in (4a–c) cannot be a constant term since it has no reference object, so it must be a variable. But as a variable it cannot be bound, unless more radical logical translations are provided. Thus, (4a–c) might conceivably be translated as, respectively:

(5) a. $\forall x[\mathrm{Donkey}(x) \to [\mathrm{Own}(\mathrm{George}, x) \to \mathrm{Feed}(\mathrm{George}, x)]]$
b. $\forall x \forall y[[\mathrm{Farmer}(x) \wedge \mathrm{Donkey}(y) \wedge \mathrm{Own}(x, y)] \to \mathrm{Feed}(x, y)]$
c. $\neg\exists x[\mathrm{Donkey}(x) \wedge \mathrm{Own}(\mathrm{George}, x)] \vee \exists x[\mathrm{Donkey}(x) \wedge \mathrm{Own}(\mathrm{George}, x) \wedge \mathrm{Feed}(\mathrm{George}, x)]$

Such translations, however, run into considerable problems. First, from a strictly linguistic point of view, there is the problem of the nonuniformity of translations, since a noun phrase like *a donkey* is to be translated as an existentially quantified expression in, for example, (4c) or *George owns a donkey*, but as a universally quantified expression in (4a,b). This would violate Russell's 'parity of form' criterion (1905,p. 483). Moreover, as was observed by Burley, (4b) allows for some farmer to own two donkeys, one that he feeds and one he does not feed, whereas (5b) is false in such a case.

Furthermore, it does not seem tenable that the pronoun *it* in (4a–c) represents a bound variable. This is so because it is typical for pronouns that do represent bound variables that they cannot be replaced with a so-called epithet pronoun, like *the great man* or *the idiot* or *the wretched animal.* Thus, in a sentence like (6a) the bound variable pronoun *they* cannot be replaced with an epithet, as in (6b), without the binding relation being destroyed:

(6) a. Some people think that *they* will get rich without working.
b. $\neq$ Some people think that *the layabouts* will get rich without working.

In (4a–c), however, the occurrences of *it* can all give way to an epithet without any referential consequences:

(7) a. If George owns a donkey he feeds *the wretched animal.*
b. Every farmer who owns a donkey feeds *the wretched animal.*
c. Either George does not own a donkey or he feeds *the wretched animal.*

This strongly suggests that the occurrences of *it* in (4a–c) are not to be analysed as bound variables but as referring expressions of some kind, even if this kind of referring expression is not known in LPC.

Thirdly, translations of the type (5a–c) fail to satisfy when intensional operators are built into the sentences in question, as in:

(8) a. If John thinks that George owns a donkey, he is certain that George feeds *it.*
b. Every farmer who is known to own a donkey, is thought to feed *it.*
c. Either John thinks that George does not own a donkey or he is certain that George feeds *it.*

If the NP *a donkey* is translated as a universally quantified expression, as in (5a,b), the meaning of the sentences in question is distorted beyond tolerable limits. If, on the other hand, existential quantification is used, scope problems arise. (5c), moreover, is questionable, as it is not simply the substitution of $\neg A \vee [A \wedge B]$ for $\neg A \vee B$, but involves the inclusion of the propositional function '$\text{Feed}(\text{George}, x)$' under the existential quantifier. (8c) shows that there are serious problems regarding the generality of this procedure.

This problem of donkey anaphora was the primary motivation behind *Discourse Representation Theory* (Kamp & Reyle 1993). A solution in terms of interpretative subdomains within the framework of *Discourse Semantics* is found in Seuren (1998*a*). Both approaches use LPC and both have extended LPC with definite descriptions and anaphoric devices, thus rejecting Russellian translations for the cases at hand and reinstating occasion sentences as elements in the semantics. Since the *logical* properties of the structures concerned do not seem to be affected by these steps in any but marginal ways, we shall leave the donkey anaphora problem undiscussed in the sequel of this paper, relegating its solution to a proper semantic theory. The emphasis of this paper is on those phenomena that are typical of occasion sentences *and lead to consequences for the logic of language*, such as presuppositions.

A similar difficulty, showing the weakness of the first assumption, concerns Russell's (1905) reduction of definite NPs to existentially quantified expressions as in (9a), translated by him as (9b):

(9) a. The present king of France is bald.
b. $\exists x[\text{Now}[\text{KoF}(x)] \wedge \text{Bald}(x) \wedge \forall y[\text{Now}[\text{KoF}(y)] \rightarrow x = y]]$

Clearly, a sentence like (10a) is not equivalent to any of its possible Russellian translations (10b–e):

(10) a. Carol thinks that there is a king of France, and she hopes that he is bald.
b. There is a king of France such that he is the only one and such that Carol thinks he is there and such that she hopes he is bald.
c. Carol thinks that there is a king of France such that she hopes that he is the only one and that he is bald.
d. Carol thinks that there is a king of France, and there is a king of France such that he is the only one and she hopes that he is bald.
e. Carol thinks that there is a king of France, and she hopes that there is a king of France such that he is the only one and is bald.

Finally, logical translations in the manner of Russell or Quine fail to solve the reference problem, which is posed by the fact that definite NPs often select their reference object in virtue of situational or world knowledge, and not on the basis of a Russellian translation as given in (9b). Under a Russellian translation, (11) is false in cases where there are several pubs. Yet for the purpose of ordinary language (11) may well be true, as long as John and Harry met in a particular pub whose identity was known and taken for granted:

(11) John and Harry met in the pub after work.

This problem is quite general. For example, in a sequence of sentences like:

(12) The book was published in 1968. The publisher was later sent to prison.

the definite NP *the publisher* must refer to the person who published the book in question in 1968, not to just any (unique!) publisher. LPC is unable to fix that reference. For it to be able to do that it must (a) be extended with a new category of intrinsically referring terms consisting of a predicate and a definite determiner, and (b) be applied first to contextually restricted cognitive structures that represent possible situations before any reference relation and hence TV can be determined.

These and similar arguments point to the following conclusions:

- Occasion sentences cannot be reduced to eternal sentences but must be recognized in their own right, both in semantics and in logic.

- If LPC is to be used for the representation of semantic content, it must be extended with at least definite descriptions and anaphoric pronouns.

- Since occasion sentences lack a TV until a key has been selected and reference values are fixed, and since these processes involve an appeal to cognition, the primary bearers of TVs are cognitive, not linguistic, structures. Linguistic utterances are TV-bearers only to the extent that they express an underlying proposition (thought).

- Only utterance *tokens*, properly embedded in a context and a situation, can be said to have a TV. Sentence *types* have logical and semantic properties, but, in principle, no TV. Eternal sentence types appear to have a TV, due to the fact that the contextual and situational embeddings required for them to have a TV are unrestricted. They therefore represent boundary cases. (This conclusion was reached earlier in Strawson (1950).)

1.2 A programme for semantics and for logic

The conclusions reached in the previous section imply a programme of research for semantic theory. First of all, a theory must be developed that specifies the cognitive structures that are taken to contain the primary bearers of TVs. This we call the THEORY OF CONTEXTUAL ANCHORING. Secondly, a THEORY OF REFERENTIAL KEYING, is needed to specify how the cognitive structures

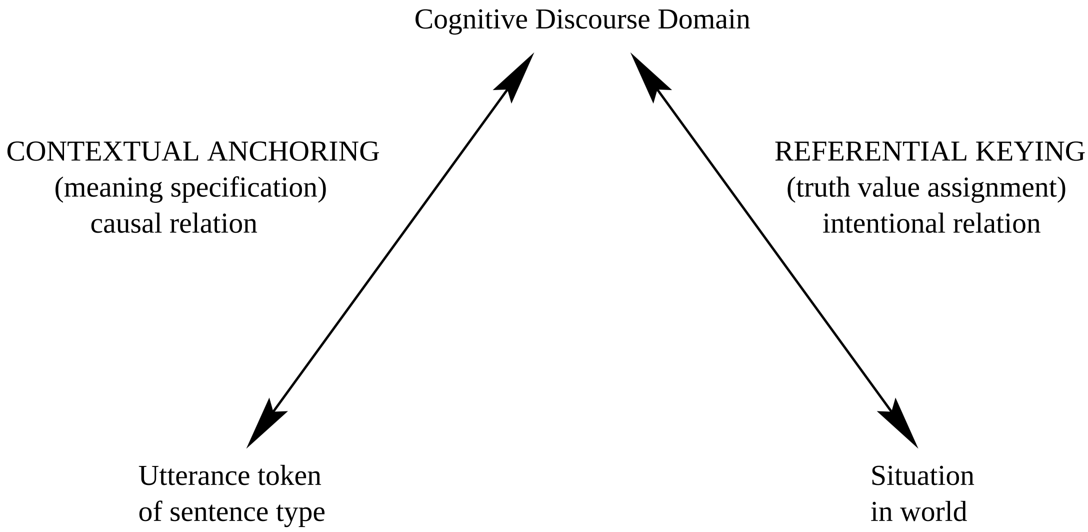


Figure 1: The triangular relation of language, mind and world

at issue, and hence the sentence tokens or utterances that express them, can be keyed to a given situation. The overall architecture into which these theories are meant to fit is schematically rendered in fig. 1.

The double arrow on the left hand side signifies a two-sided *causal* relation, in the sense that utterance tokens are produced from, or integrated into, cognitive discourse domains by means of cerebral and neuromuscular processes. The double arrow on the right hand side signifies a relation whose nature is conceptually less clear. Philosophers often speak of an *intentional* relation, which means, in principle, that the cognitive structure is *intended* to be a representation of, or 'be about', an actual situation in the world. The notion of a cognitive representation or discourse domain **D** is far from unproblematic and requires a thorough analysis of basic concepts. Yet in principle it appears to be amenable to standard methods of scientific analysis.

In essence, **D** is a structured set of structures (propositions) of the type $P(e)$, where e is an element symbol and P a property symbol, semantically defined by satisfaction conditions. If e stands for (refers to) an entity (in the widest possible sense) in the real world **W** and P stands for a well-defined property that real world entities may have, a particular proposition $P(e)$ is either true or false, according to whether the entity referred to by e does or does not have the property that P stands for. **D** may also not be about any real situation in the world at all, in which case it is not 'keyed' and has no truth value. In that case the $P(e)$-structures of **D** are, though contextually anchored, not keyed to a real world situation and are thus propositions without a TV. They are, so to speak, representations in search of a key.

Even more profound problems are raised by the notion of *intention*. To say that a proposition $P(e)$ is *intended* to be a representation of, or 'be about', an actual world situation is comprehensible in an intuitive sense, but is, as yet, not expressible in terms of causal relations and not implementable in an algorithmic model. Intentionality thus described is a mental phenomenon that still escapes the notions available in science and mathematics. For that reason it is a central and highly problematic notion in the philosophy of mind.

The intentional relation of situational keying may, however, lead to causal effects, in that the world situation may codetermine the representation(s) of the discourse domain (for example, when a speaker wants to describe a given situation), while, on the other hand, particular configurations in the discourse domain may be a determining factor in bringing about their real world counterparts (as when an order is followed).

One notes that there is no direct connection, in fig. 1, between 'Utterance token' and 'Situation'. In Ogden & Richards (1923, p. 11) a similar triangular disposition is presented for the relation between language, the mind, and the world. There a dotted line, drawn between the linguistic utterance and the world (situation), signifies a noncausal but merely 'imputed' relation determining the TV to be assigned. In the light of the arguments presented in section 1.1, it is now clear that this 'imputed' relation is based on a purely verbal notion of truth which can perhaps be made to work for eternal sentences but not for occasion sentences. It fails to take into account the fact that TVs can only be assigned to occasion sentences with the help of cognition.

Cognition, in the form of available world knowledge and discourse structure, supplies elements that are missing from the spoken signal (the utterance of an occasion sentence) but are necessary for a proper interpretation and for the assignment of a TV. These elements need not be expressed verbally, because the listener is taken to be in possession of the necessary world knowledge and to be a participant in the discourse structure at hand. Compared to a language that allows only for eternal sentences, a language that contains occasion sentences is thus seen to be superbly functional in that it saves an enormous amount of time and energy in the verbal expression of propositions.

The relation between logic, semantics and cognitive psychology is now different from what it was before. Traditionally, logic is the formal calculus of necessary consequences (entailments) given the truth of (sets of) propositions. In terms of this definition, there should be two kinds of logic, a *cognitive logic* based on thought structures, and a *verbal logic* based on linguistic structures. Since cognitive logic is still far beyond our reach, we shall, in the following, restrict ourselves to verbal logic, as is standard practice. But this means that if any verbal logic aims at handling occasion sentences, it will be unable to provide a concomitant formal theory assigning correct TVs. In other words, there will be no compositional calculus that assigns TVs to sentences in a model merely on the strength of sentence structure and model-theoretic interpretation, as is possible, in principle, for eternal sentences. Truth conditions, moreover, will have to be formulated partly in terms of parameters whose values are to be supplied by cognition.

For natural language with its occasion sentences, the process of TV-assignment is of a cognitive nature and falls, strictly speaking, within the province of cognitive psychology, outside logic and its applications in formal semantics. To the extent that established formal semantics involves a formal procedure for the assignment of TVs to natural language sentences, it must be considered not viable. In the light of the properties of occasion sentences discussed so far, it seems more appropriate to restrict semantics, in principle, to the study of the contextual anchoring properties of sentence types in discourse structures. Semantics, in other words, being the theory of linguistic comprehension, studies the building up of cognitive structures that consist of propositions each of which carries truth conditions but not necessarily a truth value. To decide how and when these conditions are satisfied in a given situation is a matter of cognitive psychology, which has, so far, not provided a formal theory.

Note that the term *proposition* will be used, from now on, for subject-predicate structures that are well-anchored in context and thus contribute to a meaningful text. If a proposition is also properly keyed to a situation, it will have a TV, but it need not have one to be meaningful.

For an uttered sentence token S to have a TV it must satisfy two global conditions: (a) S must be contextually anchored, and (b) S must be keyed to a situation in the world. When only condition (a) is fulfilled but not condition (b), S is part of a meaningful text thought up by an author, but its TV is irrelevant. Or, in Frege's words:

> Why is the thought not enough for us? Because, and to the extent that, we are concerned with its truth value. This is not always the case. In hearing an epic poem, for instance, apart from the euphony of the language we are interested only in the sense of the sentences and the images and feelings thereby aroused. The question of truth would cause us to abandon aesthetic delight for an attitude of scientific investigation. Hence it is a matter of no concern to us whether the name *Odysseus*, for instance, has reference, so long as we accept the poem as a work of art. It is the striving for truth that drives us always to advance from the sense to the reference.
>
> *Frege (1892, p. 33) translation by Max Black in Geach & Black (1970, p. 63)*

When not even condition (a) is fulfilled, S is unanchored and hence uninterpretable, but still meaningful in the general sense that it may play a role in the building up of cognitive structures consisting of propositions and possibly carrying a TV. Although these conditions apply to occasion sentences in particular, we shall henceforth speak of sentences in general, since eternal sentences are considered boundary cases, whose anchoring and keying conditions are always met. In section 1.3 condition (a) is discussed. Condition (b) is discussed in section 1.4.

1.3 Contextual anchoring and presuppositions

For a sentence S to be contextually anchored (or be part of a coherent discourse) it must satisfy at least the following necessary conditions:

a. Every definite term in S has a unique denotation (address) in the discourse domain **D**.

b. All presuppositions of S are incremented in **D** before S.

Following a by now widely accepted view, we consider a discourse domain **D** to be a structured representation of an ordered set of sentences. A **D** must contain at least a number of 'addresses' representing possible objects (singular or plural, natural individuals or reifications). Every new well-anchored sentence is *incremented* in **D** in that the new information provided by S is added to **D**. The precise format in which one may best take this to be done is not our concern here. Two main strategies present themselves: either the predicate label expressing the property assigned by S is added to the appropriate addresses that correspond to the definite terms in S, or the appropriate address labels are added to the predicate label. A combination of both is also thinkable. New addresses are introduced by means of existential quantification.

Condition (a) requires that **D** be structured in such a way that each definite term in S corresponds uniquely to an address in **D**. For definite descriptions (e.g. *the house*) this means that the determiner *the* seeks the unique address in **D** that is characterized by the predicate *house*. Definite pronouns need to find a proper antecedent, i.e. an address recently activated by explicit mentioning. If a definite description fails to find an address in **D**, the missing address can be supplied on grounds of knowledge-based inference, as is demonstrated in (12) above for the definite description *the publisher*. For pronouns this is, normally speaking, not possible (try to read (12) with *he* for *the publisher*).

Condition (b) is to do with presuppositions. *We consider a presupposition to be a proposition P implied in, and structurally recoverable from, a sentence S (its 'carrier sentence') in such a way that P must precede S in **D** for S to be interpretable.* A presupposition P of a carrier sentence S thus poses a condition on **D** for the meaningfulness or interpretability of S or the simple negation of S.

Four main categories of presupposition can be distinguished:

i. Existential presuppositions, as in ('≫' stands for 'presupposes'):

(13)	John took his son to the Zoo.	≫	John exists; John has a son; there is a Zoo

ii. Factive presuppositions (presupposing the truth of the *that*-clause), as in:

(14)	John noticed that he was getting wet.	≫	John was getting wet

iii. Categorial presuppositions, implied in the meaning of the predicate, as in:

(15) a.	David is divorced.	≫	David was married before
b.	David has stopped beating his dog.	≫	David has beaten his dog before

iv. Remainder category, to do with focusing strategies and the particles *only* and *even*, as in:

(16) a.	JOHN didn't laugh, HARRY did.	≫	Somebody laughed
b.	Only John laughed.	≫	John laughed

It makes sense, however, to assume that for all categories of presupposition the semantic source of the presuppositions of a sentence S is, in principle, located in the satisfaction conditions of the highest predicate of S (see section 2.3.1 below). In light of the observations made in 2.3.1 below, it seems advisable, if not inevitable, to distinguish between two classes of satisfaction conditions, the *preconditions*, whose nonsatisfaction results in RADICAL FALSITY, and the *update conditions*, whose nonsatisfaction results in MINIMAL FALSITY. Satisfaction of all conditions yields truth. From a purely logical point of view, presupposition is then a lexically driven entailment, induced by lexical preconditions. The reduction to lexical satisfaction conditions is straightforward for the categories (i)–(iii). For category (iv) it is possible only if, at a level of semantic analysis, particles like *only* or *even* are considered focusing predicates and a specific focusing predicate is assumed for contrastive accents and other contrastive or emphatic focusing strategies such as clefting. This aspect of presuppositional analysis, however, will not be gone into further in the present context.

Since a sentence S_P (i.e. S presupposing P) requires P to be incremented in $\mathbf{D}$ before S, a speaker asserting S_P cannot be committed to the truth of S without also being committed to the truth of P, on analytical grounds, i.e. grounds of meaning. It follows that if $S \gg P$, then $S \vDash P$. Moreover, since under normal conditions the contextual anchoring conditions of a sentence S are identical to those of its negation *not-S*, a speaker asserting *not-S_P* cannot be committed to the truth of *not-S* without also being committed to the truth of P. Hence, if $S \gg P$, then *not-S* $\vDash P$. We thus formulate as a logical condition for presupposition (applicable under the default conditions):

(17) If $S \gg P$, then $S \vDash P$ and *not-S* $\vDash P$.

But this poses a problem for the logic of language, since in standard logic, if $S \vDash P$ and *not-S* $\vDash P$, P must be a necessary truth. In language, however, presuppositions are as contingent as any other proposition. This problem is solved in section 2.3.3 below, where the trivalent propositional calculus PPC_3 is presented.

It is is important to realize that a description of the logical properties of presupposition does not automatically give a semantic definition. On the contrary, a sound logic is a necessary but not a sufficient property for a sound natural language semantics (see section 2.3.5). It is thus possible for a pair of sentences A and B to satisfy all the logical conditions of the semantic relation of presupposition without the one presupposing the other. Conversely, however, if $A \gg B$, then A and B must show the appropriate logical properties defined in PPC_3. The semantic dimensions that go beyond logic are not explored here.

It must be noted that existential presupposition differs from denotational anchoring (condition (a)), in that the latter is required by definite terms looking for a unique address in $\mathbf{D}$, whereas the former is induced by the predicate in question, which may or may not require real existence for one or all of its term referents. Thus, a sentence like *John is talking about the Abominable Snowman* requires the availability of a unique address for the description *the Abominable Snowman* (condition (a)), but it does not presuppose the existence of such a creature, since the predicate *talk about* does not require real existence of its object term referent (it is intensional with respect to its object term). For $\mathbf{D}$ this implies that the expression *the Abominable Snowman* may seek its denotation address in some intensional subdomain representing somebody's belief or story, in case the main (or truth) domain lacks an appropriate address.

Since presuppositions are structurally recoverable from their carrier sentences, it is, in principle, not necessary to present presuppositions explicitly, in the form of actual utterance tokens. For any S_P, it is sufficient to pronounce only S, since P can be, and very often is, cognitively 'slipped in' when S is processed. This process is called ACCOMMODATION or POST HOC INSERTION (PHI). The process of PHI is blocked only in cases where it would result in an inconsistent $\mathbf{D}$ or where implicit relations lack sufficient cognitive backing. The latter is illustrated in, for example,

(18) When John entered the house, the corridor started to pray.

Supposing that John and the house are already 'in the story', the corridor is easily supplied by PHI, since it is normal for houses to have corridors and one may expect a listener to know that. But it is not normal for corridors to pray, and any such relation will have to be explained first for an utterance of (18) to be interpretable. Failing such an explanation, (18) is not interpretable.

Most normal texts contain a multitude of presuppositions 'slipped in' by PHI. Given the relatively large amounts of time and energy involved in the actual production and comprehension of utterance tokens, the mechanism of PHI constitutes a powerful energy-saving device. It is important to realize, however, that this device is crucially dependent on the cognitive ability to detect inconsistencies and on available background knowledge.

The presuppositions of a sentence may be parallel or stacked. For example, a sentence like:

(19) John realizes that Mary's best friend is divorced.

has the parallel presuppositions 'There is a person called "John"' and 'Mary's best friend is divorced'. The latter, however, again presupposes 'Mary's best friend was married before', which presupposes 'Mary has a best friend', which again presupposes 'There is a person called "Mary"'. These presuppositions thus stand in the structural relationship to each other shown in fig. 2 (where '$A \rightarrow B$' means 'A is presupposed by B'). All these presuppositions are recoverable from the

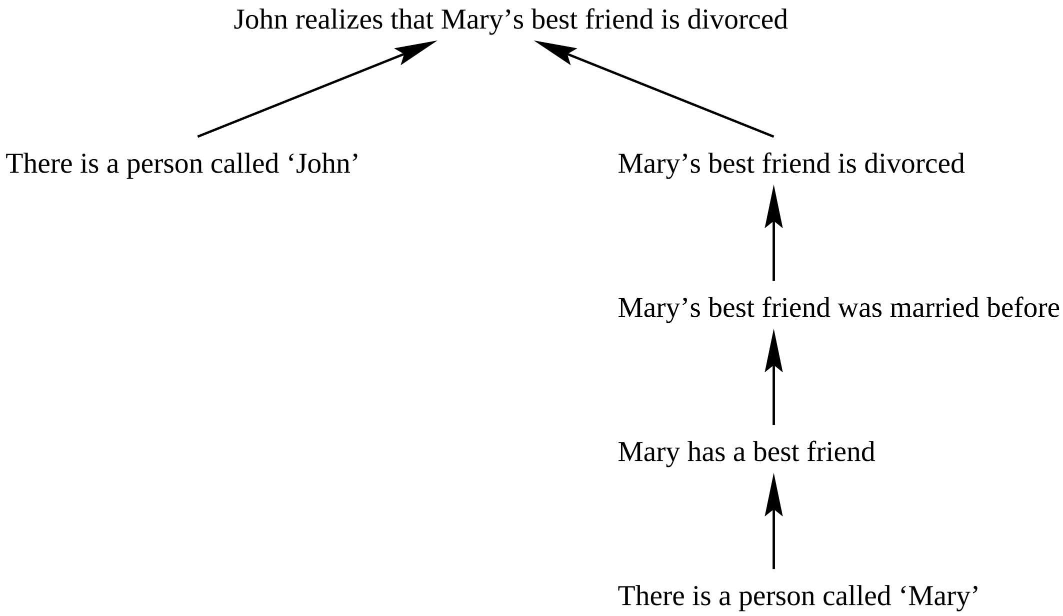


Figure 2: The presuppositional structure of (19)

carrier sentence (19) and can thus be 'slipped in' by means of PHI, in the proper order. We remark here that PHI inserts all hereditary presuppositions of the sentence. For the example sentence (19) this implies that all sentences in fig. 2 that are below sentence (19) are inserted by PHI. This conforms with the fact that the presuppositions of the presuppositions of a sentence S are themselves presuppositions of S.

Apart from a few late 19th century admonitions (e.g Sidgwick 1895) to the effect that context and discourse should be considered essential factors in any adequate semantic theory of natural language, the first modern proposals to this effect go back to the early 1970s, in particular Seuren (1972, 1975), Stalnaker (1973), Isard (1975). They were soon followed by a spate of theories and proposals that share the property of being incremental (and thus tend at least to consider a rehabilitation of occasion sentences) but differ widely in other respects, notably McCawley (1979), Van den Auwera (1979), Ballmer (1979), Lewis (1979), Wunderlich (1979), Karttunen & Peters (1979), Gazdar (1979), Kamp (1981), Heim (1982, 1983), Barwise & Perry (1983), Fauconnier (1985), Landman (1986), Burton-Roberts (1989), Groenendijk & Stokhof (1991), Kamp & Reyle (1993), and many others. While many of these do reinstate definite descriptions in the (explicit or implicit) logical analysis, thus opening the way towards satisfying condition (a) mentioned at the outset of this section, only very few take condition (b), which is about presuppositions, into account. And to the extent that they do, only a few consider the logical aspects of presupposition, the others being restricted either, rather myopically, to so-called projection phenomena (which fall outside any logical analysis) or to largely informal pragmatic analyses, or both. The only remaining approach that considers both presuppositions and their strictly logical aspects, Burton-Roberts (1989), is only remotely incremental and, moreover, just like all other approaches, fails to take into account the specific observations presented in 2.3.1 below and published earlier in Seuren (1985, 1988) and elsewhere (for a detailed critique of Burton-Roberts 1989 see Seuren 1990). It is precisely these facts that call for a specific trivalent logic with two kinds of falsity (PPC_3). A similar conclusion was reached in Dummett (1973, p. 421) on comparable but not identical grounds (see below), but neither Dummett's nor Seuren's argument was ever acknowledged in the literature on presuppositions. Therefore, in spite of the many interesting aspects of the literature at hand, none of it is relevant for the present more restricted purpose, which is to reinstate occasion sentences and to investigate their logical and mathematical foundations in a way that takes account of all relevant facts.

1.4 Situational keying and reference fixation

Every S has to be *keyed* to a situation for it to have a TV. A *key* consists in the specification of where to look for verification or falsification. No theory has been developed so far to account for either the speaker's intentional keying in to a particular situation in the real world $\mathbf{W}$, or the listener's adequate picking up of the intended key. For the listener this appears not to be a strictly compositional process, but rather a matter of hypothesis and approximation. Unrestricted truth is anyway not a sufficient criterion. If it were, a complete fleshing out of all presuppositions of an occasion sentence by means of PHI, as illustrated in fig. 2, would be sufficient to provide all occasion sentences with a TV without any intentional keying. It would then, for example, be sufficient for the truth of (19) that there be persons called 'John' and 'Mary', respectively, that Mary have a best friend who was married before but is now divorced, while John realizes all that. But although there may be many situations in the actual world that satisfy these conditions, this does not make (19) true. The truth or falsity of (19) requires a prior intentional focusing on a particular situation shared by speaker and listener. As a matter of principle, TVs are predicated on prior keying, and this fact must be taken into account in any theory of truth and meaning, as well as in an adequate logic of natural language sentences. Formal philosophical, semantic and logical theories of natural languages are thus subservient more to formal analyses of cognition than to mathematical logic. The role of the latter is still highly relevant, but more restricted than is standardly thought.

It is now clear that straightforward-looking instances of eternal sentences, such as *There isn't a person called 'John'* or *Everybody wants lower taxes*, can be true even if there is, somewhere in the big wide world, a person called 'John' or someone of whom it is not true to say that he or she wants lower taxes. To say that the truth or falsity of such statements is *pragmatically* restricted to certain situations may well be correct, under an appropriate definition of the term 'pragmatic', but it is not very enlightening unless the full consequences are drawn for the theory of truth and meaning, and for a proper logic of natural language sentences.

It is probably correct to say that the fixation of *reference* comes after the fixation of a key, i.e. the intentional focusing on a specific situation. This appears from the fact that key-restricted truth is sometimes used as a means for the fixing of reference. This phenomenon, described in Seuren (1985, pp. 459–464) as 'nonspecific reference', is illustrated by a sentence like:

(20) John owns a dog, and it bit him.

uttered with respect to a situation where a person called 'John' owns two dogs, one that bit him and one that did not. In that situation (20) is true, and it is so in virtue of the fact that the definite term *it* automatically selects the dog that satisfies the conditions of the predicate *bit him*, so that the second conjunct is true. That is, the reference of *it* (or of *John's dog*) is made dependent on the truth of the proposition 'it (John's dog) bit him'. This means that the sentence:

(21) John owns a dog, and it did not bite him.

is likewise true in the same situation, because in this case the reference object of *it* is the dog that did not bite him. This fact is remarkable because truth is here used as a criterion for the fixing of reference given a situational key. For the second conjuncts of (20) and (21) to be true it is sufficient for there to be, in the situation at hand, a dog that did, or did not, bite John, respectively.

This puzzling fact was noticed by Walter Burley, as was shown in connection with example (3) above, and is specifically discussed in Geach (1969) (though again without attribution). Beyond that, however, it has escaped the attention of modern philosophy, probably because it has been assumed that Geach's solution to the problem is adequate. Geach's solution amounts to 'translating' (20) and (21) not as a conjunction of two propositions, i.e. as $A \wedge B$, but as, respectively

(22) a. $\exists x[\mathrm{Dog}(x) \wedge \mathrm{Own}(\mathrm{John}, x) \wedge \mathrm{Bite}(x, \mathrm{John})]$
b. $\exists x[\mathrm{Dog}(x) \wedge \mathrm{Own}(\mathrm{John}, x) \wedge \neg\mathrm{Bite}(x, \mathrm{John})]$

so that inconsistency is avoided. It was shown, however, in Seuren (1977) that this solution is inadequate since it does not apply to cases where intensional operators are involved, as in:

(23) a. John must have owned a dog, and it may have bitten him.
b. John must have owned a dog, and it cannot have bitten him.

Both (23a) and (23b) may be true at the same time, provided John owned at least two dogs. But Geach's solution does not apply, due to scope problems. If *it* is taken to represent a variable bound by an existential quantifier $\exists x$, as in (22a,b), then the operators 'possible' in (23a) and 'not-possible' in (23b) must be in the scope of $\exists x$. But $\exists x$ itself is in the scope of the necessity operator *must*, in the normal interpretation of (23a,b). It follows that *may* and *cannot* must likewise be in the scope of *must*, which is clearly not what these sentences mean. It is, therefore, impossible to bind *it* in the cases quoted, which makes Geach's solution invalid for these cases. This conclusion is reinforced by the observation that the pronoun *it* in (20) and (21) can be replaced with an epithet, as in :

(24) a. John owns a dog, and *the animal* bit him.
b. John owns a dog, and *the animal* did not bite him.

which, as we have seen, appears to be impossible for pronouns representing bound variables.

The consequences of the phenomenon of nonspecific reference are startling. First, the Language of Predicate Calculus must be extended at least with pronominal definite terms that are not bound variables. Secondly, and more importantly, even if that is done, the standard model-theoretic calculus by which TVs are computed on the basis of the extensions of terms and predicates in the model cannot be upheld, since here the extension of some terms is determined by the assumed value 'true' for the proposition at hand, which would make the procedure circular.

The phenomenon of nonspecific reference shows that keying and reference fixation are cognitive processes in a game of hypothesis and approximation, and cannot be part of logical model theory. In fact, standard model-theoretic semantics, to the extent that it takes keying and reference relations into account (toy models usually do), simply takes these for granted. But this means that the empirical question of how language users come to understand and interpret their sentences remains fundamentally unsolved in model-theoretic semantics. The Quinean programme of reformulating occasion sentences as eternal sentences is an attempt at circumventing this problem, but, as has been shown, to no avail. We must conclude that natural language semantics is basically different from what is called 'semantics' in logic.

2 The logic of occasion sentences

2.1 The logic of occasion sentences is restricted to prior selection of key and reference

It is now clear that a formal theory of entailments, i.e. a logic, of natural language sentences is predicated on the prior selection of a key **K** and of reference relations in **K**. In its simplest form,

$\mathbf{K}$ is defined by a set $\mathbf{I}$ of individuals in $\mathbf{W}$, within frames of time and place. A discourse is said to be *about* $\mathbf{K}$. A new sentence in a discourse may open up a new $\mathbf{K}$, in which case the discourse is about more than one $\mathbf{K}$. Normal discourses are about sets of $\mathbf{K}$s forming a *hyperkey*. In the present context hyperkeys will be left out of account, and only simple $\mathbf{K}$s will be considered.

A key $\mathbf{K}$ realizes a particular actual state of affairs or situation $\mathbf{s_a}$, but other situations $\mathbf{s}$ might have occurred in $\mathbf{K}$, depending on what relations obtain in $\mathbf{I}$. We say that $\mathbf{K}$ is a set of situations $\mathbf{s}$, one of which is the actual situation $\mathbf{s_a}$.

If a natural language $\mathbf{L}$ is considered to be a set of sentences, not all sentences of $\mathbf{L}$ are interpretable given some $\mathbf{K}$. Only the sentences in a subset $\mathbf{L_K}$ of $\mathbf{L}$ will be interpretable given $\mathbf{K}$. There is as yet no formal method for delimiting $\mathbf{L_K}$ given some $\mathbf{L}$ and given some $\mathbf{K}$ (hardly surprising when one realizes the neglect of occasion sentences in modern logic and semantics). Sentences not belonging to some $\mathbf{L_K}$ have no truth value and are, therefore, not objects in any logical calculus.

Each sentence $A \in \mathbf{L_K}$ is associated with the set of situations $\Sigma \subseteq \mathbf{K}$ in which A is true, or the Σ-SPACE of A, also written as $/A/$. Every Σ-space is a *possible fact*. When, for some A, $/A/ = \mathbf{K}$, A is necessarily true in $\mathbf{K}$. When $/A/ = \emptyset$, A is necessarily false in $\mathbf{K}$. We call the Σ-space of a sentence A, or $/A/$, the *extension* of A. A sentence $A \in \mathbf{L_K}$ is true just in case $\mathbf{s_a} \in /A/$, and false just in case $\mathbf{s_a} \notin /A/$.

2.2 Applications of Boolean algebra to standard propositional calculus

In his famous article (1892), Frege decided to apply the distinction between intension and extension, which had so far been restricted to predicates, also to sentences. He stipulated that the extension of a sentence A, or $[\![A]\!]$, should be the *truth value* of A, whereas the intension of A should be the *thought* underlying A in the minds of language users. His reason for taking TVs as extensions of sentences was one of convenience. According to Frege, the TV of a sentence can be computed compositionally from the extensions of its component parts (1892,p. 33-4). Thus, if the extension of a sentence is taken to be its TV, there is a compositional calculus to compute the extension of a sentence on the basis of the extensions of its parts and nothing else. The fact that such a calculus is not available for the intension (underlying thought) of a sentence makes this extensional calculus all the more valuable (it is the basis of Montague's programme of 'extensionalisation of intensions'). It has been shown above that it is not correct to say that the TV of a sentence can be computed compositionally from the extensions of its parts, not even if one limits oneself (which Frege did not do) to eternal sentences, since the satisfaction conditions of predicates often require an appeal to world knowledge. But Frege did not take such niceties into account.

A further convenience for Frege was the fact that if TVs are sentence extensions, Boolean algebra computes the truth functions. All that is needed is to define 'truth' as the value of Boolean 1, and 'falsity' as the value of Boolean 0. Negation ($\neg$) is now interpreted as Boolean complement, conjunction ($\wedge$) as Boolean multiplication, and disjunction ($\vee$) as Boolean addition. This is the origin of the widespread convention to denote truth with the symbol '1', and falsity with '0'.

The propositional truth-functional operators now compute as follows. For any sentences A, B:

- A is true iff $[\![A]\!] = 1$; A is false iff $[\![A]\!] = 0$.
- $\neg A$ is true iff $[\![A]\!] = 0$; $\neg A$ is false iff $[\![A]\!] = 1$. That is, $[\![\neg A]\!] = \overline{[\![A]\!]}$.
- $[\![A \wedge B]\!] = [\![A]\!] \cdot [\![B]\!]$ and $[\![A \vee B]\!] = [\![A]\!] + [\![B]\!]$.

This gives the classical truth tables of fig. 3. However, although this gives the correct computations for the standard truth functions, it remains unclear what is meant when one says that a sentence A is true, or false. All one can say, with Frege, is that a true sentence refers to the VERUM or 'the True', whereas a false sentence refers to the FALSUM or 'the False'. However, as a basis for a philosophically sophisticated theory of truth (and meaning), this Fregean application is

$$
\begin{array}{c|c}
\llbracket A \rrbracket & \llbracket \neg A \rrbracket \\
\hline
1 & 0 \\
0 & 1
\end{array}
\qquad
\begin{array}{cc|cc}
\llbracket A \wedge B \rrbracket & & \llbracket B \rrbracket & \\
 & \cdot & 1 & 0 \\
\hline
\llbracket A \rrbracket & 1 & 1 & 0 \\
 & 0 & 0 & 0
\end{array}
\qquad
\begin{array}{cc|cc}
\llbracket A \vee B \rrbracket & & \llbracket B \rrbracket & \\
 & + & 1 & 0 \\
\hline
\llbracket A \rrbracket & 1 & 1 & 1 \\
 & 0 & 1 & 0
\end{array}
$$

Figure 3: Frege's application of Boolean algebra to bivalent propositional calculus

unsatisfactory and thus open to revision. It requires that the truth values, being extensions, be considered part of the world with respect to which sentences (propositions) are true or false. But the VERUM and the FALSUM are hardly defensible as elements in any ontology, a fact widely recognized in model-theoretic semantics but left unremedied.

There is, however, a different though, as far as standard bivalent calculus is concerned, logically equivalent notion of sentence extension, sketched in section 2.1 above and based on the notion of Σ-space. It was said there that the extension of a sentence A is a possible fact, or the set of situations in **K** in which A is true. This we have decided to call the Σ-space of A or $/A/$. The idea originates with Boole (1847, pp. 49-50), but was never fully elaborated. Kneale & Kneale (1962, p. 43) speak of a 'perhaps more interesting' development. To the extent that one understands Boole's few remarks on the matter, it seems that he had in mind an interpretation where Boolean '1' is the algebraic expression for the universe **U**, or the set of all possible situations, of which the actual situation $\mathbf{s_a}$ is one. '0' is the algebraic expression for the empty set or $\emptyset$. For any sentence A of **L**, the extension of A, let us say again $/A/$, is the set of situations in which A is true. Apparently, Boole did not realize that most sentences of any natural language are occasion sentences, which means that they are not true or false *per se* but only when properly anchored and keyed. This makes the notion of 'set of possible situations in which a sentence A is true' incoherent. Yet, if this complication is disregarded by always applying the logical calculus *modulo* **K**, Boole's notion provides an alternative to Frege's notion of sentence extension, which is logically equivalent as long as the logic is kept strictly bivalent.

Van Fraassen (1971, pp. 88ff) was the first to provide a formal elaboration of Boole's idea, still in terms of an unrestricted universe **U**, i.e. the set of all possible situations, without any contextual or keying restrictions. For Van Fraassen, a situation is defined by a valuation, i.e. an assignment of truth values to all sentences of a language **L**. If **L** contains n logically independent sentences, then the number of valuations for **L** is 2^n, with the two values T ('true') and F ('false'). The Σ-space (for Van Fraassen the *valuation space*) of a sentence A, or $/A/$, is the set of valuations in which A gets the value T. Clearly, if $A \vDash B$, then $/A/ \subseteq /B/$. If $A \vDash B$, any valuation where A is valued T and B is valued F is *inadmissible*, in Van Fraassen's terms.

This allows for a Boolean interpretation of standard propositional calculus. Let each constant term in the algebra stand for the Σ-space of a sentence in the language. Variables ranging over terms thus stand for arbitrary Σ-spaces. For any necessarily true sentence N_t in **L**, $/N_t/ = \mathbf{U}$ (read '1'). For any necessarily false sentence N_f, $/N_f/ = \emptyset$ (read '0'). $\overline{/A/}$ is the set of valuations (Σ-space) in which A is false. When A is true, the valuation v_a describing the actual situation is a member of $/A/$: $v_a \in /A/$. When A is false, $v_a \notin /A/$ and $v_a \in \overline{/A/}$. It follows that $/\neg A/ = \overline{/A/}$. Thus, when A is false, $v_a \in \overline{/A/}$, or $v_a \in /\neg A/$. We now define:

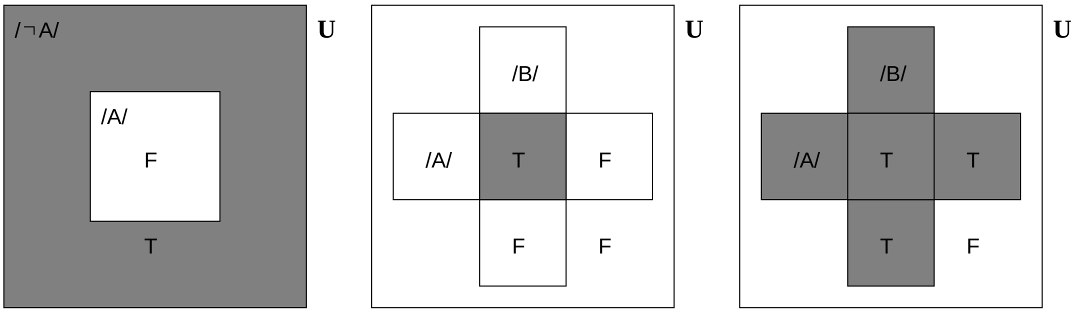
$$/A \wedge B/ = /A/ \cdot /B/ \qquad \text{and:} \qquad /A \vee B/ = /A/ + /B/$$

This likewise gives the classical truth tables in fig. 4, with T for 'true' and F for 'false'.

$$
\begin{array}{c|c}
/A/ & /\neg A/ \\
\hline
\mathsf{T} & \mathsf{F} \\
\mathsf{F} & \mathsf{T}
\end{array}
\qquad
\begin{array}{cc|cc}
/A \wedge B/ & & /B/ & \\
 & \cdot & \mathsf{T} & \mathsf{F} \\
\hline
/A/ & \mathsf{T} & \mathsf{T} & \mathsf{F} \\
 & \mathsf{F} & \mathsf{F} & \mathsf{F}
\end{array}
\qquad
\begin{array}{cc|cc}
/A \vee B/ & & /B/ & \\
 & + & \mathsf{T} & \mathsf{F} \\
\hline
/A/ & \mathsf{T} & \mathsf{T} & \mathsf{T} \\
 & \mathsf{F} & \mathsf{T} & \mathsf{F}
\end{array}
$$

Figure 4: Σ-space application of Boolean algebra to bivalent propositional calculus

The truth tables are demonstrated more clearly by means of set-theoretic diagrams (fig. 5). In these diagrams the Σ-spaces and the corresponding values T and F are positioned in such a way that the truth tables can be read directly from the diagrams. The same method is followed in the figs. 9, 10 and 13 below.



$/\neg A/ = \overline{/A/}$ $\quad$ $/A \wedge B/ = /A/ \cdot /B/$ $\quad$ $/A \vee B/ = /A/ + /B/$

Figure 5: Set-theoretic interpretation of bivalent propositional calculus

In the following section it will be shown that the logic of natural language must be at least trivalent, as it distinguishes two different kinds of falsity. In the light of that distinction, Frege's notion of TVs as sentence extension and his use of Boolean 1 for truth and Boolean 0 for falsity cannot be upheld, simply because Boolean 0 does not allow for internal distinctions. If, however, truth and falsity are treated in terms of Σ-spaces, there is no problem, since Σ-spaces, being sets, allow for further internal distinctions. From now on, therefore, we shall use Van Fraassen's Σ-space application of Boolean algebra as the formal foundation of propositional calculus, with one important difference. Since it makes no sense to say of occasion sentences that they are true or false *per se*, for any given situation, without specifying how they are anchored and keyed, we shall not speak of the universe **U** of all possible situations, but rather of the key **K** of all possible situations in which the sentences of $\mathbf{L_K}$ are true or false. The 'universe of discourse', in other words, is not the unfathomable totality of all possible situations ('worlds'), with all the conceptual, logical and ontological problems that come with it, but the rather more manageable set of possible states of affairs within the restricted part of the world focused upon by means of the intentional mental act of keying. Apart from that, Van Fraassen's analysis can be maintained in its entirety, since the underlying mathematics remains the same.

2.3 The logic of presupposition

2.3.1 Presupposition requires trivalence

In this section an empirical argument is proposed to the effect that the logic of natural language cannot be bivalent but must at least be trivalent, with two different kinds of falsity. Before the argument can be presented, the notion of bivalence has to be stated with some precision. The Aristotelian Principle of Bivalence, also known as the Principle of the Excluded Third (PET), applies first and foremost to the Aristotelian theory of truth as correspondence. Its application to logic is secondary. For Aristotle, truth and falsity are properties of propositions expressed in sentences, in such a way that PET holds. PET consists of the following two independent subprinciples:

i. Principle of Complete Valuation: all propositions always have a truth value.

ii. Principle of Binarity: there are exactly two truth values, 'true' and 'false'; there are no values in between, and no values outside 'true' and 'false'. The Principle of Binarity comprises the Principle of the Excluded Middle (PEM), which says only that there are no values between 'true' and 'false', and says nothing about possible values beyond simply 'true' and 'false'.

The Principle of Complete Valuation holds trivially if one follows the tradition, which says that to have a truth value is a defining feature of a proposition. Then, obviously, it makes no sense to speak of propositions without a truth value. Under our definition, however, of a proposition as a subject-predicate structure that is contextually anchored, it makes a great deal of sense. For now the Principle of Complete Valuation implies that keying is not necessary and that all anchored sentences are automatically keyed. It has been argued that this must be considered incorrect.

The Principle of Binarity, on the other hand, can be rejected in a number of ways. One may, for example, wish to reject the Principle of the Excluded Middle or PEM, and maintain that the opposition between true and false is not, as Aristotle insisted it was, absolute, like that between locked and unlocked, but gradable, like that between polite and impolite. An elaboration of this notion leads to what is known as 'fuzzy logic' (Zadeh 1975), which allows for an infinite number of values between 'true' and 'false'. When all intermediate values are taken together as one intermediate third value, the result is a trivalent logic with an intermediate value between 'true' and 'false', such as the trivalent logic devised by Kleene (1938, 1952) (although Kleene did not set up his trivalent logic with this purpose in mind). Such logics defy PEM and hence the Principle of Binarity.

A different way of rejecting the Principle of Binarity, mentioned earlier in section 1.3, consists in distinguishing different kinds of falsity. An example may illustrate this. Suppose a quiz master asks the question:

> Which of these four was the youngest president ever of the United States:
> Reagan, Jefferson, Kennedy or De Gaulle?

The correct answer is, of course, 'Kennedy. But of the three incorrect answers, one is somehow more incorrect than the other two. The answer *De Gaulle was the youngest president ever of the US* is somehow 'worse' than the answers that mention Reagan or Jefferson, because De Gaulle does not even fulfill the preliminary condition of having been president of the US. It is possible, or thinkable, to exploit this difference theoretically by distinguishing two kinds of satisfaction conditions, the PRECONDITIONS and the UPDATE CONDITIONS. The extension of the predicate *be the youngest president of the US* can thus roughly be specified as follows:

$$[\![\text{be the youngest president of the US}]\!] \quad = \quad \{x : x \text{ is or was president of the US} \mid \text{there is no } y \text{ such that } y \text{ is or was president of the US and } y \text{ is or was younger than } x\}$$

The conditions between the colon and the upright stroke are the preconditions. Those after the upright stroke are the update conditions. Failure to satisfy the preconditions results in RADICAL FALSITY (F_2). Failure to satisfy the update conditions results in MINIMAL FALSITY (F_1). Satisfaction of all conditions results in TRUTH (T). The preconditions, moreover, determine the PRESUPPOSITIONS of the sentence in question. In this perspective, the sentence *De Gaulle was the youngest president ever of the US* presupposes that De Gaulle was president of the US. Since this presupposition is false, the sentence is radically false.

The argument here is that the behaviour of sentence negation in natural language, in connection with presuppositions, makes it mandatory to distinguish between minimal falsity and radical falsity in the way indicated. The first proposal to this effect was made in Dummett (1973, p. 421), also on grounds of presupposition and negation, though more from a philosophical than from an observational angle. (Dummett also considers the possibility of two kinds of truth, a suggestion that should be taken seriously but is not elaborated here.) An actual trivalent propositional calculus (PPC_3) was provided in Seuren (1985, 1988).

Since, under the Principle of Binarity, all situations (whether in $\mathbf{U}$ or in $\mathbf{K}$) are such that either A or $\neg A$ is true, it follows that when $A \vDash B$ and also $\neg A \vDash B$, B must be a necessary truth (true in all situations of either $\mathbf{U}$ or $\mathbf{K}$). In empirical terms this means that if it can be established that in natural language a sentence A as well as its negation *not-A* both entail a sentence B which is not a necessary truth (in $\mathbf{U}$ or in $\mathbf{K}$), then natural language *not* cannot correspond to the bivalent negation operator $\neg$ of standard propositional calculus. If *not* is to be rendered in the logic of language as a truth-functional operator, room must be created for a third option, besides standard truth and falsity, the 'third' excluded by PET.

The point now is that there are many sentence pairs $(A, \textit{not-}A)$ in natural language, such that both A and *not-A* entail a sentence B which is not a necessary truth in any sense of the term. Examples are given in (25)–(31) below (similar observations are presented and commented upon in much grater detail in Seuren 1985, 1988, 2000). In all such cases the shared entailment B is a presupposition of A as well as of *not-A*.

(25)	a.	All children laughed.	$\models$	there were children
	b.	Not all children laughed.	$\models$	there were children
(26)	a.	Only the children laughed.	$\models$	the children laughed
	b.	Not only the children laughed.	$\models$	the children laughed
(27)	a.	The BUTLER killed Jack.	$\models$	someone killed Jack
	b.	The BUTLER didn't kill Jack (JOE did).	$\models$	someone killed Jack
(28)	a.	It was the BUTLER that killed Jack.	$\models$	someone killed Jack
	b.	It wasn't the BUTLER that killed Jack.	$\models$	someone killed Jack
(29)	a.	Who killed Jack was the BUTLER.	$\models$	someone killed Jack
	b.	Who killed Jack wasn't the BUTLER.	$\models$	someone killed Jack
(30)	a.	That Joe died surprised Susan.	$\models$	Joe died
	b.	That Joe died didn't surprise Susan.	$\models$	Joe died
(31)	a.	She doesn't mind that Joe has left.	$\models$	Joe has left
	b.	She DOES mind that Joe has left.	$\models$	Joe has left

The sentence pairs (25–31) distinguish themselves from the majority of pairs $(A, \textit{not-}A)$ in that normally a sentence *not-A* allows for the cancelling of presuppositional entailments if the negation word *not* is given heavy accent and the whole sentence is placed under an echo-intonation. Thus, in (32) the presuppositional implication that there is a king of France can be cancelled under the intonational conditions mentioned. Yet there remains a more or less strong suggestion or invited inference that there is a king of France, an inference mistaken by many for an entailment:

(32) The present king of France is not bald.

In his famous (1905), Russell maintained that (32) does not entail that there is a king of France, although it suggests it. His solution consisted in analysing or 'translating' (32) in two different ways:

(33) a. $\neg\exists x[\mathrm{Now}[\mathrm{KoF}(x)] \wedge \mathrm{Bald}(x) \wedge \forall y[\mathrm{Now}[\mathrm{KoF}(y)] \to x = y]]$
b. $\exists x[\mathrm{Now}[\mathrm{KoF}(x)] \wedge \neg\mathrm{Bald}(x) \wedge \forall y[\mathrm{Now}[\mathrm{KoF}(y)] \to x = y]]$

(33a) is the ordinary full sentential negation of (9b), his translation of (9a), *The present king of France is bald*, whereas in (33b) the negation is restricted to the propositional function 'Bald(x)'. For reasons best known to natural language speakers, Russell says, (33b) appears to be preferred and (33a) appears to be the marked case. Why speakers should have this preference is left open by Russell. That question was taken up in modern pragmatics (which has, however, failed to provide an answer).

Leaving aside the question of whether Russell's 'translations' (33a,b) are justifiable, we must admit that he was right in claiming that (32) is open to two interpretations, one that saves the presupposition of (9a), and one that cancels it. If this were the case for all negative sentences in natural languages, then there would indeed be some point in saying that full sentential negation, as in (33a), cancels all entailments, so that standard propositional calculus can stand. It is found, however, Seuren (1985, pp. 118–238) that there are many cases where the reading expressed in (33a) is not possible. These are, first, all cases where the sentence negation is not in its 'canonical' position, i.e. in construction with the finite verb, as in (25b) and (26b). Such 'out-of-place' negations, apparently, have no choice but to preserve all presuppositional entailments. Yet the only possible translation for these sentences places the negation at the top: in all these cases the negation is full sentence negation, even though the presuppositional entailments are preserved. Since this is not possible in standard bivalent logic, something has to be done about the logic.

This fact is illustrated neatly by the following three English sentences (where the exclamation mark indicates communicational incoherence):

(34) a. He did not only sell his collection of rare books. He only sold his first edition of Milton.
b. ! Not only did he sell his collection of rare books. He only sold his first edition of Milton.
c. ! He not only sold his collection of rare books. He only sold his first edition of Milton.

The sentence *He only sold his collection of rare books* presupposes that he sold his collection of rare books and asserts that he sold nothing else. This presupposition can be cancelled in (34a), where *not* is in construction with the finite verb *did*. However, in (34b,c) *not* is in different positions, allowed for by the grammar of English, and here the presupposition cannot be cancelled, as is borne out by the incoherence of (34b,c). Nor is it possible to 'translate' them in such a way that *not* is no longer a full sentential negation.

Returning now to (25a,b), we see that the presupposition that there were children is maintained under sentence negation, apparently because *not* does not occur in the canonical position for sentence negation. One realizes, of course, that in standard Predicate Calculus (25a) does not entail that there were children (though (25b) does on account of the fact that 'not all' is equivalent to 'some not', which has existential import). Yet standard Predicate Calculus does an injustice to natural language in this respect, as was also recognized by Strawson (1952) and by Aristotle, whose Predicate Calculus had existential import as a valid inference schema (the 'subaltern'). As is well known, the Aristotelian inference from 'all' to 'some' leads to logical disaster when empty sets are quantified over, but it is all right as long as empty sets are avoided. In other words, Aristotelian Predicate Calculus *presupposes* the nonemptiness of the sets quantified over. This means that Aristotle implicitly, and no doubt without realizing it, not only took proper anchoring and keying for granted in his Predicate Calculus, but also limited it to situations where presuppositions are fulfilled. Under these restrictions, Aristotelian Predicate Calculus is sound.

The examples (27–29) are to do with focusing in terms of three syntactically different types of topic-comment structure. Apparently, for reasons not yet worked out (but surely to do with the principles of coherent discourse) focusing structures cannot give up their presuppositions under negation.

Example (30) involves the predicate *surprise* which is factive with regard to its subject clause (i.e. the truth of the subject clause is presupposed). As long as the subject clause stays in the syntactic position for subjects, the factive presupposition cannot be shed under negation. By way of contrast, consider:

(35) a.	It surprised Susan that Joe died.	$\models$	Joe died
b.	It did not surprise Susan that Joe died.	$\not\models$	Joe died

where (35b) no longer entails that Joe died, since now the 'radical' interpretation of *not* is possible.

In (31) we have to do with the NEGATIVE POLARITY ITEM *mind*, which requires either a negative context or contrastive accent, as in (31b), for the sentence to be grammatical. Negative polarity items, likewise, do not allow for presuppositions to be dropped under negation.

Cases like (25–31) show that sentence negation does not *per se* cancel presuppositional entailments, but clearly preserves them in certain sentence types. This fact shows that the classical bivalent paradigm cannot be upheld, unless some external remedy is found. In the logic-based theory of model-theoretic semantics it has been hoped, for the past quarter century, that *pragmatics* would provide such an external remedy. Yet no such remedy has been provided. That being so, we feel justified in saying that it makes sense to look for ways to extend standard bivalent logic in such a way that the observations made above are accounted for in logical terms. The obvious solution would then seem to consist in adding a third truth value and making the logic trivalent.

2.3.2 Kleene's trivalent calculus

A first notable attempt to do just that was made in Kleene (1938, 1952), mentioned earlier. Kleene's trivalent calculus is widely used in logic-oriented presupposition research (e.g. Blau 1978).

Yet closer analysis reveals that it is unfit for that purpose, although it does serve the different purpose of accounting for phenomena to do with transitional values between true and false.

What Kleene had in mind was a logical account of sentences containing nonreferring terms, i.e. terms whose proper semantic function is to refer to a world entity whereas the world does not contain such an entity, precisely as in Russell's famous sentence (9a). Such 'undefined terms' would make the sentence have the TV 'undefined' or 'u'. This trivalent calculus, with the values T, u, and F, works according to the truth tables shown in figs. 6 and 7. One sees that under negation T and F 'toggle' in the classical way, but that u is unaffected by negation, that under conjunction ($\wedge$) F takes precedence over all other values, and u over T, whereas under disjunction ($\vee$) T takes precedence over all other values, and u over F. In fig. 6 this leads to the fan-like structure in the tables for $\wedge$ and $\vee$, with T as the root of the fan for $\wedge$, and F for $\vee$. In the equivalent tables of fig. 7 where u is ordered as the third value, after T and F, the fan-like structure has disappeared. We shall see in a moment that this is significant: for a proper Σ-space interpretation the fan-like disposition of the values is mandatory.

A	$\sim A$
T	F
u	u
F	T

$A \wedge B$		B: T	u	F
A	T	T	u	F
	u	u	u	F
	F	F	F	F

$A \vee B$		B: T	u	F
A	T	T	T	T
	u	T	u	u
	F	T	u	F

Figure 6: Truth tables of Kleene's trivalent propositional calculus

A	$\sim A$
T	F
F	T
u	u

$A \wedge B$		B: T	F	u
A	T	T	F	u
	F	F	F	F
	u	u	F	u

$A \vee B$		B: T	F	u
A	T	T	T	T
	F	T	F	u
	u	T	u	u

Figure 7: Truth tables of Kleene's trivalent propositional calculus

This logic maintains all axioms of classical bivalent logic with the negation operator '$\sim$' for standard '$\neg$', except $\vDash A \vee \sim A$. In particular, De Morgan's Laws apply unchanged:

(36) a. $\sim(A \wedge B) \equiv \sim A \vee \sim B$
b. $\sim(A \vee B) \equiv \sim A \wedge \sim B$

That the Kleene calculus fails to account for presuppositions appears from the following. It is assumed, in accordance with all theories of presupposition, that (37) is a defining logical property of the presupposition relation. (Since Kleene provides no operator yielding truth when $v[A] = u$ (A is valued u), we introduce the operator 'u' and define: $v[uA] =$ T iff $v[A] = u$ and $v[uA] =$ F otherwise.)

(37) If $A \gg P$, then $A \vDash P$ and $\sim A \vDash P$ and $(\sim P \vee uP) \vDash uA$.

Moreover, in any reasonable notion of presupposition, it must be assumed that:

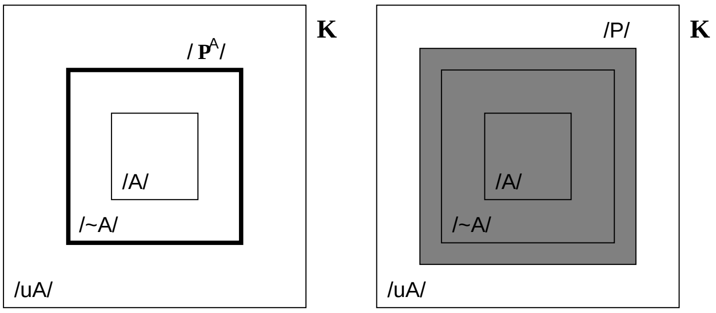
(38) $A_C \wedge B_D \gg C \wedge D$, where A and B are logically independent ('X_Y' : 'X presupposing Y').

For neither $A_C \wedge B_D$ nor $\sim(A_C \wedge B_D)$ can be contextually anchored unless the presuppositions of A and B, i.e. C and D, respectively, are part of the preceding discourse (see 1.3). This means that $A_C \wedge B_D$, provided A_C and B_D are well-keyed, can only have the values T or F if both C and D are true. Here, the Kleene calculus poses a problem. Take a situation where C is true and D is false and A_C is false (and, of course, B_D has the value u, since its presupposition D is false).

Now $C \wedge D$ has the value F, which should make it necessary for $A_C \wedge B_D$ to have the value u. Yet, with F for A_C and u for B_D, the Kleene tables give F for $A_C \wedge B_D$, and not the required value u.

The deeper reason why the Kleene calculus fails in this respect becomes clear in the Σ-space interpretation. Since, in general, if $A \models B$, then $/A/ \subseteq /B/$, it follows from (37) that if $A \gg P$, then $/A/ \subseteq /P/$ and $/\sim A/ \subseteq /P/$ and $\overline{/P/} \subseteq /uA/$. In fact, if $\mathbf{P}^A$ stands for the conjunction of all presuppositions of A, then $/\sim A/ \cup /A/$ must equal $/\mathbf{P}^A/$. We call $/\mathbf{P}^A/$ the PRESUPPOSITIONAL SUBUNIVERSE of A.

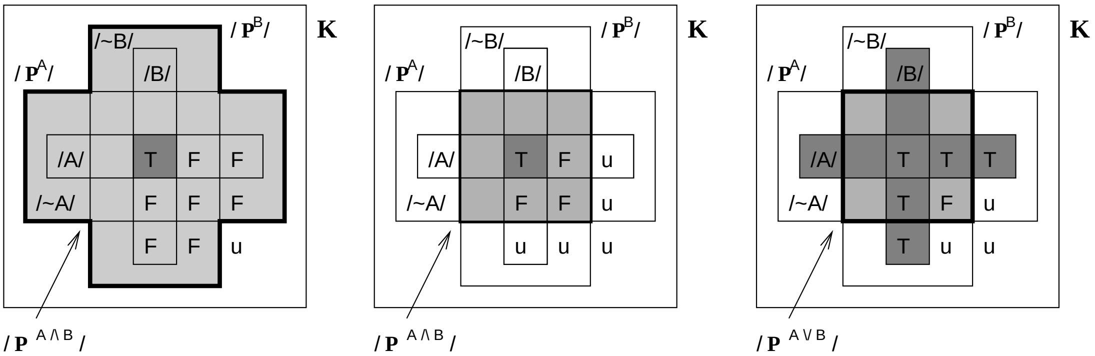
If Kleene's calculus is to account for the presupposition relation, $/\sim A/$ must be defined as $/\mathbf{P}^A/ - /A/$, as in fig. 8 (left), where $/\mathbf{P}^A/$ (the area within heavy lines) equals $/A/ \cup /\sim A/$. Fig. 8 (right) shows that both A and $\sim A$ entail their presupposition P ($/P/$ is represented by the dark grey area).



$/\sim A/ \cup /A/ = /\mathbf{P}^A/$ $\qquad$ $/A/ \subseteq /P/$ and $/\sim A/ \subseteq /P/$

Figure 8: If $A \gg P$, then $/A/ \subseteq /P/$ and $/\sim A/ \subseteq /P/$

However, it is now impossible to set out the Σ-spaces for $\sim(A \wedge B)$ and $\sim(A \vee B)$ in such a way that De Morgan's Laws apply under the Kleene truth tables. De Morgan's Laws require that $/\sim(A \wedge B)/ = /\sim A/ \cup /\sim B/$ and $/\sim(A \vee B)/ = /\sim A/ \cap /\sim B/$.

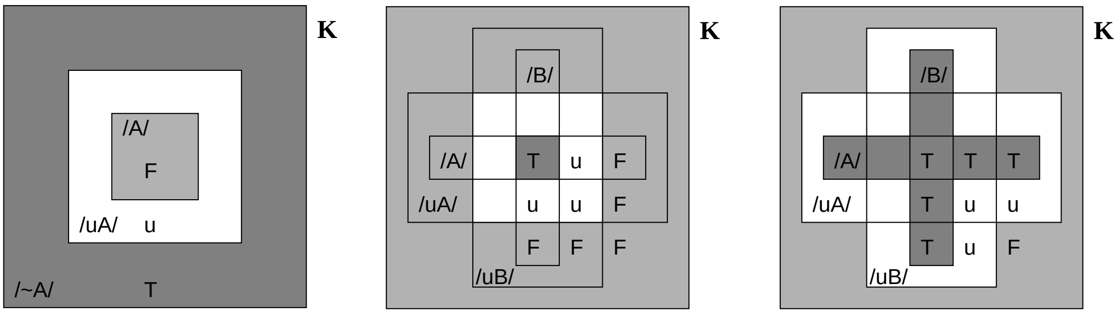


$/A \wedge B/ = /A/ \cap /B/$ $\qquad$ $/A \wedge B/ = /A/ \cap /B/$ $\qquad$ $/A \vee B/ = /A/ \cup /B/$
$/\sim(A \vee B)/ = /\sim A/ \cap /\sim B/$

Figure 9

In fig. 9 we have tried to picture the situation where the requirements of De Morgan's Laws are fulfilled, given the definition of presuppositional $\sim$ as in fig. 8. The conjunction has been represented twice, once with $/\mathbf{P}^{A\wedge B}/ = /\mathbf{P}^A/ \cup /\mathbf{P}^B/$ and once with $/\mathbf{P}^{A\wedge B}/ = /\mathbf{P}^A/ \cap /\mathbf{P}^B/$. In either case, however it is not so that $/\sim(A \wedge B)/ = /\sim A/ \cup /\sim B/$, quite apart from the fact that the truth tables do not correspond. Moreover, the diagram for $/\sim(A \vee B)/$ in fig. 9 (right) violates (37), since nontruth of $\mathbf{P}^{A\vee B}$ does not automatically result in the value u for $A \vee B$. (The dark grey areas contain the situations in $\mathbf{K}$ that produce T, the light grey areas those that produce F, and the white areas those that produce u, for $\sim A$, $A \wedge B$, and $A \vee B$, respectively. The areas within heavy lines represent $/\mathbf{P}^{A}/$, $/\mathbf{P}^{A\wedge B}/$, and $/\mathbf{P}^{A\vee B}/$, respectively.) Therefore, De Morgan's Laws cannot be made to hold in the Kleene calculus under a presuppositional interpretation.

The only way to satisfy Kleene's calculus in a Σ-space interpretation is to leave out the notion of presupposition and consider the value u as a transition between T and F, as in fig. 10. Now De Morgan's Laws hold and the right truth tables result, but the presupposition relation cannot be expressed. The only way to define $/\mathbf{P}^{A\wedge B}/$ is to take in fig. 10 (middle) the union of all the non-white areas. Then, however, $/\mathbf{P}^{A}/ \cap /\mathbf{P}^{B}/ \subsetneq /\mathbf{P}^{A\wedge B}/$, where one would expect these to be equal. But even if we take this inequality for granted, we still cannot accept the definition of $/\mathbf{P}^{A\wedge B}/$, because if A has the value F and B has the value u, then $\sim(A \wedge B)$ has the value T, whereas the conjunction of the presuppositions of A and B has the value F, violating (37) and (38). For that reason we have said, in section 2.3.1, that the Kleene calculus seems appropriate as a logical account of a violation of PEM, if the value u is taken to incorporate all intermediate values between T and F. Note, incidentally, that while in fig. 10 $/\sim(A \wedge B)/ = /\sim A/ \cup /\sim B/$ and $/\sim(A \vee B)/ = /\sim A/ \cap /\sim B/$, the analogous equations with u for $\sim$ are not valid. De Morgan's Laws thus do not hold for the operator u.



$/\sim A/ = /\mathbf{P}^{A}/ - /A/$ $\quad$ $/A \wedge B/ = /A/ \cap /B/$, $/\sim(A \wedge B)/ = /\sim A/ \cup /\sim B/$ $\quad$ $/A \vee B/ = /A/ \cup /B/$, $/\sim(A \vee B)/ = /\sim A/ \cap /\sim B/$

Figure 10

2.3.3 The trivalent presuppositional calculus PPC_3

In order to satisfy the logical conditions (37) and (38) of the presupposition relation, it is necessary to define, for A_B, where B is the conjunction of all presuppositions of A, a presuppositional subuniverse or subkey $\mathbf{P}^{A}$ such that $/\sim A/ = /\mathbf{P}^{A}/ - /A/$ and $/\mathbf{P}^{A}/ := /B/$.

Three values are distinguished: T, F_1 and F_2, and two complementary negations, the minimal presupposition-preserving negation $\sim$ and the *radical* presupposition-cancelling negation $\simeq$. (The classical bivalent negation $\neg$ has been thrown in for good measure.) We call $/\sim A/$ the INNER COMPLEMENT of A, and $/\simeq A/$ the OUTER COMPLEMENT of A.

This gives the truth tables shown in fig. 11. (The implication is left undefined in PPC_3, because conditional sentences in natural language are clearly not truth-functional but imply a modal notion of necessity which cannot be expressed by means of a truth table. But if one wishes, an implication of the form $A \rightarrow B$ can be defined as '$(\sim A \vee \simeq A) \vee B$', which reduces this implication to the classical implication. And analogously for the bi-implication $A \equiv B$.)

For PPC_3 conjunction, F_2 takes precedence over the other values and F_1 over T. For disjunction, T takes precedence over the other values and F_1 over F_2. Note that, for any proposition A, $(\sim A \vee \simeq A) \equiv \neg A$ (with the classical bivalent negation $\neg$). PPC_3 is, therefore, equivalent to classical bivalent propositional calculus provided only $\neg$ is used as negation. (In particular, $\neg(A \wedge B) \equiv \neg A \vee \neg B$ and $\neg(A \vee B) \equiv \neg A \wedge \neg B$.) The negations $\sim$ and $\simeq$ are called *specific* negations, because they turn one specific kind of falsity into truth. $\neg$ is a nonspecific negation in PPC_3.

PPC_3 can be extended to PPC_n, with $n - 1$ kinds of falsity. Conjunction always selects the highest degree of falsity and truth only if there is no falsity at all. Disjunction always selects

A	$\sim A$	$\simeq A$	$\neg A$
T	F_1	F_1	F_1
F_1	T	F_1	T
F_2	F_2	T	T

$A \wedge B$		B: T	F_1	F_2
A	T	T	F_1	F_2
	F_1	F_1	F_1	F_2
	F_2	F_2	F_2	F_2

$A \vee B$		B: T	F_1	F_2
A	T	T	T	T
	F_1	T	F_1	F_1
	F_2	T	F_1	F_2

Figure 11: Truth tables of PPC_3

truth over falsity, and lower degrees of falsity over higher degrees. For each $\overset{i}{\sim} A$ ($1 \leq i < n$), T and all values F_j ($j < i$) are converted to F_1, F_i is converted to T, and all values F_k ($k > i$) are left unchanged. For PPC_4 this is shown in fig. 12. Note that $\neg$ is still the disjunction of all specific negations. An interesting corollary is that a propositional calculus, defined in terms of $\{\neg, \wedge, \vee\}$, may have any number of truth values. However, without further specific negations for specific complements, all distinctions between kinds or degrees of falsity are vacuous. Economy then requires that all values $\neq$ T be united into one value for falsity.

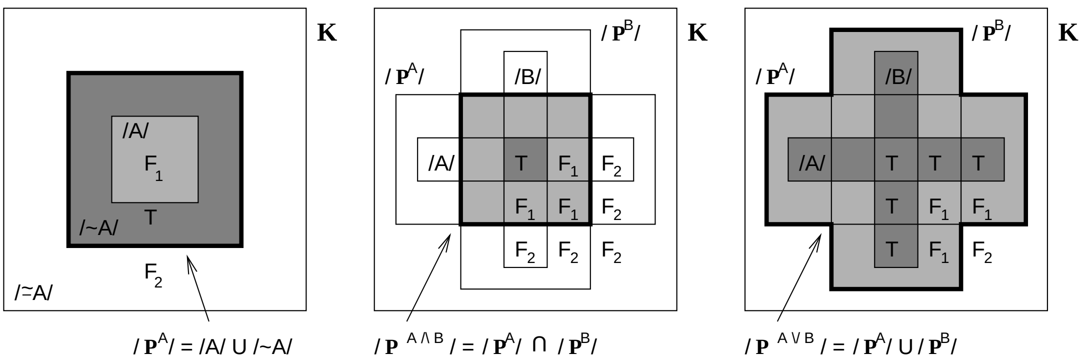
A	$\overset{1}{\sim} A$	$\overset{2}{\sim} A$	$\overset{3}{\sim} A$	$\neg A$
T	F_1	F_1	F_1	F_1
F_1	T	F_1	F_1	T
F_2	F_2	T	F_1	T
F_3	F_3	F_3	T	T

$A \wedge B$		B: T	F_1	F_2	F_3
A	T	T	F_1	F_2	F_3
	F_1	F_1	F_1	F_2	F_3
	F_2	F_2	F_2	F_2	F_3
	F_3	F_3	F_3	F_3	F_3

$A \vee B$		B: T	F_1	F_2	F_3
A	T	T	T	T	T
	F_1	T	F_1	F_1	F_1
	F_2	T	F_1	F_2	F_2
	F_3	T	F_1	F_2	F_3

Figure 12: Truth tables of PPC_4

In a Σ-space interpretation, PPC_3 is represented as in fig. 13. This is an exact parallel of fig. 10, except that the Σ-spaces of $\sim A$ and $\simeq A$ (or uA in fig. 10) have changed positions: in fig. 10, $/\!\sim A/$ is the outer complement and $/uA/$ the inner complement of A, whereas in fig. 13 $/\!\sim A/$ is the inner complement, and $/\!\simeq A/$ the outer complement of A. This means that in PPC_3 De Morgan's Laws hold for the operator $\simeq$ (and, of course, for $\neg$), but not for $\sim$ (unless the value F_2 is disregarded).



$/\!\sim A/ = /\mathbf{P}^A/ - /A/$ $\quad$ $/A \wedge B/ = /A/ \cap /B/$ $\quad$ $/A \vee B/ = /A/ \cup /B/$

Figure 13

Note that an eternal sentence, and thus without any presuppositions, can still be regarded, from a strictly logical point of view, as presupposing all necessary truths. An eternal sentence has no outer complement and cannot have the value F_2. Its inner complement is the classical complement in $\mathbf{K}$, and standard bivalent logic applies. Thus, a sentence with the internal structure $A \wedge B_A$ can be read as $A_{\mathbf{K}} \wedge B_A$. Its Σ-space $/A_{\mathbf{K}} \wedge B_A/ = /B_A/$, and $/\mathbf{P}^{A_{\mathbf{K}} \wedge B_A}/ = /\mathbf{P}^{A_{\mathbf{K}}}/ \cap /\mathbf{P}^{B_A}/ = \mathbf{K} \cap /A/ = /A/$.

A further important point is the following. In section 1.3 above, the logical condition (17) was formulated for the presupposition relation, saying that if $A \gg B$, then $A \models B$ and *not*-$A \models B$. It was stipulated there that this condition does not define presupposition but is merely a necessary condition, since there may be cases where (17) is satisfied but where we do not want to speak of presupposition. This occurs in particular under the operator $\wedge$, and specifically with conjunctions of the type $A \wedge B_A$, which are very frequent in language use, because they are *informative* in the sense that $/B_A/ \subset /A/$.

It follows from PPC_3 that both $A \wedge B_A \models A$ and $\sim(A \wedge B_A) \models A$. Yet we do not want to say that $A \wedge B_A \gg A$. The reason is that in language a sequence A *and* B is processed in any current discourse domain $\mathbf{D}$ as the increment of A followed by the increment of B. A temporal order is thus involved in the processing of A and B, which cannot be expressed in the static truth-functional system PPC_3. This temporal order is manifest in the presupposition relation in the manner shown in fig. 14. Let A be a sentence without presuppositions, so that $/\mathbf{P}^A/ = \mathbf{K}$. The left diagram

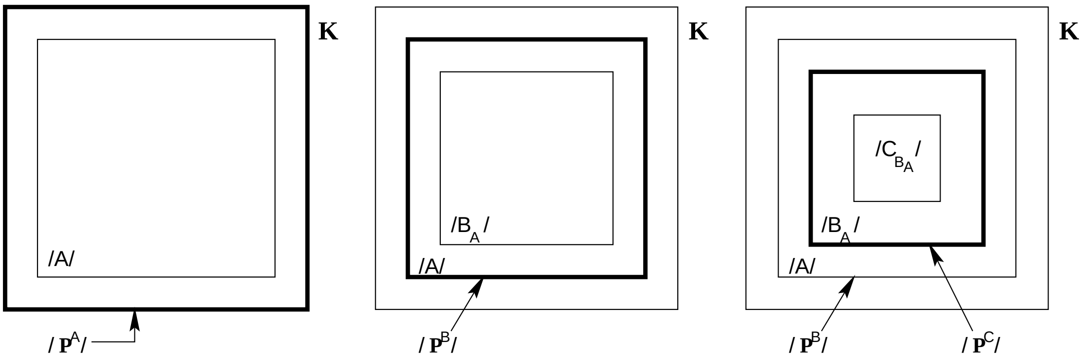


Figure 14: Subsequent processing of A, B_A and C_{B_A}

shows $\mathbf{K}$ after A has been incremented, or added, in $\mathbf{K}$. The middle diagram shows $\mathbf{K}$ after the incrementation of B_A, with $/\mathbf{P}^B/ = /A/$, and the right diagram shows the situation after the addition of C_{B_A}, now with $/\mathbf{P}^C/ = /B/$. That is, after each successive incrementation the space within which the minimal negation operates gets more restricted, and previous presuppositional subuniverses are cancelled. Since linguistic *and* is an operator signalling a new incrementation, the use of a minimal negation over a conjunction of the type A *and* B_A is logically undefined: in the middle diagram of fig. 14, the minimal negation operates within $\mathbf{K}$ for A but within $/A/$ for B_A. For that reason a structure like $\sim(A \wedge B_A)$, though logically sound in PPC_3, has no logically equivalent translation in natural language. A sentence like

(39) He did not marry a princess and divorce her after one year.

does not correspond to the logical structure $\sim(A \wedge B_A)$. In fact, no logical translation of that sentence is available at present. This being so, we do not want to say that A *and* B_A presupposes A, whereas we do want to say that the logically equivalent B_A does.

One might consider a system where $\mathbf{K}$ and $/A/$ in the right diagram of fig. 14 are defined as 'higher order' subuniverses delimiting inner complements under 'higher order' negations. In that case the logic would fluctuate between 2 and n values according to the number $n + 2$ of stacked presuppositions, perhaps as shown for PPC_n above. But such a system would not model natural language, which does not have a corresponding system of unlimited 'higher order' negations.

So we are faced with a situation where, although $/\mathbf{P}^C/ = /B/$, the inner complement of $/B/$ is different from the inner complement of $/\mathbf{P}^C/$, since the inner complement of $/B/$ is delimited with regard to $/\mathbf{P}^B/ = /A/$, while that of $/\mathbf{P}^C/$ is delimited with regard to $\mathbf{K}$, without any intervening presuppositional subuniverse. Phrased in other terms: a presuppositional proposition has no presuppositions itself. For the extension of the proposition $\mathbf{P}^{\mathbf{P}^C}$, this means that $/\mathbf{P}^{\mathbf{P}^C}/ = \mathbf{K}$, even though $/\mathbf{P}^C/ = /B/$ and $/\mathbf{P}^B/ = /A/$. The consequences for the calculus of presuppositional subuniverses are explained in section 3 below.

2.3.4 Kleene's calculus and PPC_3 combined into PPC_3-K

PPC_3 and the Kleene calculus are compatible and can be combined into PPC_3-K. The Kleenean value 'u' between two values x and y is interpreted as 'vague between x and y'. Since PPC_3 contains three values, T, F_1 and F_2, PPC_3-K contains two values u: u_1 and u_2. The truth tables of PPC_3-K are as in fig. 15. The Σ-space interpretation of PPC_3-K is as in fig. 13 above, but with

A	$\sim A$	$\simeq A$
T	F_1	F_1
u_1	u_1	F_1
F_1	T	F_1
u_2	u_2	u_1
F_2	F_2	T

$A \wedge B$		B: T	u_1	F_1	u_2	F_2
A	T	T	u_1	F_1	u_2	F_2
	u_1	u_1	u_1	F_1	u_2	F_2
	F_1	F_1	F_1	F_1	u_2	F_2
	u_2	u_2	u_2	u_2	u_2	F_2
	F_2	F_2	F_2	F_2	F_2	F_2

$A \vee B$		B: T	u_1	F_1	u_2	F_2
A	T	T	T	T	T	T
	u_1	T	u_1	u_1	u_1	u_1
	F_1	T	u_1	F_1	F_1	F_1
	u_2	T	u_1	F_1	u_2	u_2
	F_2	T	u_1	F_1	u_2	F_2

Figure 15: Truth tables of $\mathsf{PPC}_3 - K$

the boundary lines between $/A/$ and $/\sim A/$, and between $/\sim A/$ and $/\simeq A/$ blurred or replaced with a transitional area. The value u_1 stands for the transitional area between $/A/$ and its inner complement $/\sim A/$. This value is assigned when A is neither clearly true nor clearly minimally false. The value u_2 stands for the transitional area between $/\mathbf{P}^A/$ and $/\simeq A/$. It is assigned when a presupposition of A is neither clearly true nor clearly (minimally) false. In either case the minimal negation $\sim$ has no effect. The radical negation $\simeq$, which says that A suffers from presupposition failure, yields (minimal) falsity when A is true, minimally false or somewhere in between, and gives minimal undefinedness (u_1) when a presupposition of A is (radically) undefined.

2.3.5 The relation between semantics and logic

It is probably correct to say that the tables of fig. 15, as far as they go, do partial justice to the logic of natural language. They certainly provide an answer to the ancient paradoxes of the Heap 'Sorites' and the Horns (Seuren 1998*b*, p. 427). Yet standard bivalent logic remains privileged, in that it is adequate for languages without vague predicates and whose anchoring and keying conditions are automatically fulfilled so that presuppositions are either absent or irrelevant. One such language is the language of mathematics, but many formal or technical uses of natural language satisfy these conditions as well.

However, whether the tables of fig. 15 also do *semantic* justice to natural language as used under normal conditions is another matter. From a logical point of view, natural language is more complex than standard bivalent logic, due, in part, to its anchoring and keying conditions. But besides this greater complexity, which is partly caught in the tables mentioned, there is also the fact that logic and semantics are less closely related than is widely assumed in formal semantics.

Even when a correct and adequate logic of natural language is available, it does not follow automatically that the logical elements (quantifiers, connectives) as described in the logic of language provide a correct semantic analysis of their corresponding elements in language. Several aspects play a central role in semantics but are absent from a logical analysis, which is concerned solely with the preservation of truth through sets of sentences. In Seuren (2000) it is argued that speech act quality is an essential aspect of semantic theory, unjustly assigned to pragmatics in standard formal semantics. It is argued there that the propositional connectives, including negation, are more adequately accounted for in terms of different forms of speaker's commitment, and not in terms of truth functions.

It is likewise argued there that the logical consequences of the fact that natural language happily mixes object language and metalanguage, apparently without the risk of paradoxes, have been unjustly neglected in standard formal semantics and in the philosophy of language. The linguistic counterpart of radical negation is richer than its logical representative $\simeq$, in that it has a specific metalinguistic function (Horn 1985), which is not captured by its logical definition. This aspect, which is analysed in detail in Seuren (2000), cannot be further elaborated here.

The logic of natural language, in other words, is considered to be a metaphysically necessary epiphenomenal aspect of the elements, structures and processes at issue. Questions of this nature are profound and far from easy to understand, and it cannot be the purpose of the present paper to provide a final answer. What we wish to achieve here is, more modestly, to bring these questions to the fore and show their importance. Formal semantics has, on the whole, overlooked or neglected these questions. They are, however, highly relevant, if only because the machinery of logic, no matter how enlightening and inspiring from a variety of points of view, can hardly be considered to provide or constitute a realistic hypothetical reconstruction of the mental structures and processes involved in the understanding and interpretation of linguistic utterances.

In the following section, the mathematical properties of PPC_3 are investigated, not because PPC_3 is regarded as a semantic theory, but rather because it is essential for any semantic theory that proof be given of the mathematical soundness of the logic emerging from it.

3 The Boolean foundation of PPC_3

3.1 Noncompositionality

The fact that PPC_3 is representable by means of a set-theoretic diagram as in fig.13 means that it must have a Boolean foundation. Since this is not provided in the logical or mathematical literature, it is developed in the present section. It must likewise be possible to develop a Boolean foundation for the Kleene calculus and for PPC_3-K. In order not to complicate matters unduly, this is not attempted here: we shall limit ourselves to PPC_3.

We anticipate immediately that one major problem in the mathematical theory of presuppositional logic is the noncompositionality of the system. By *compositionality* we mean here the admissibility of substitution of equal terms inside a context. Let $\mathcal{C}(a_1, \dots, a_n)$ be a context in which the terms $a_1, \dots, a_n$ occur. The *substitution property* states that we can substitute equal terms in place of $a_1, \dots, a_n$, i.e. that if $b_1 = a_1, \dots, b_n = a_n$, then $\mathcal{C}(b_1, \dots, b_n) = \mathcal{C}(a_1, \dots, a_n)$. If the substitution property holds the equality '$=$' is said to be a *congruence*. This property fails for PPC_3 because two Σ-spaces may be equal without having the same inner complements, as was explained at the end of section 2.3.3 above. We can intuitively explain this phenomenon by saying that the equality '$=$' is blind to presuppositions and can see only extensions of propositions. This gives us the idea of defining a new equality '$\equiv$' that is able to see presuppositions as well. That is, $a \equiv b$ means that not only the extensions of a and b are the same, but also those of their presuppositions. In the next sections we will formulate and study the system PPC_3 with the weak equality '$=$' and as from section 3.5 we will study a compositional version, PPC_3^c, which uses the strong equality '$\equiv$'.

Consider a Boolean system where a term a stands for the Σ-space of some sentence A of a language L, i.e. as $/A/$. The principal innovation with regard to standard Boolean algebra consists in the introduction of an operator '1' such that 1_a represents the presuppositional subuniverse $/\mathbf{P}^A/$ of A. The symbol '1' is here used as a unary operator that, when applied to the Boolean term a representing the extension $/A/$ of a proposition A, delivers the Boolean term 1_a representing $/\mathbf{P}^A/$.

The choice of the symbol '1' for the operator at hand has been deliberate. It underlines the fact that 1_a is interpreted as a presuppositional subuniverse for the corresponding sentence A. It may look as if the symbol '1' is used ambiguously as (i) a Boolean constant (a constant in all Boolean systems) and (ii) an operator over terms yielding terms. We can, however, generalize the notion of 1 as an operator in such a way that the Boolean constant 1 is seen as a special case of the operator 1. The operator 1 is thus taken to be basic, the constant 1 being derived from it. We do this by defining 1 (without argument) as the common value for all 1_{1_a} for any term a. Moreover, $1_1 = 1$ (equation ($e21$) in Proposition 3.7 below) and $1_0 = 1$ (axiom ($D4$) in Definition 3.3).

There is a deeper significance to this. The fact that the constant 1 is now derived from the noncompositional function 1 makes an interpretation of 1 in a system, such as the system of Σ-spaces, less absolute. It is no longer necessarily the unwieldy 'universe' of all that is or may be the case, but rather a 'universe' or key in so far as it is relevant to a given discourse. It is now also possible to have different 'universes' or keys side by side in a hypersystem of systems running in parallel. It would seem that, in principle at least, this opens new possibilities for a more adequate logic to model discourses.

As was shown at the end of section 2.3.3, the operator 1 is noncompositional, since it is possible for two sentences in natural language to have identical Σ-spaces yet to differ in their inner complements. That is, we do not have in general $a = b \to 1_a = 1_b$. A concrete example will illustrate this.

Consider the propositions expressed in the following sentences, corresponding exactly to A, B and C, respectively, in fig. 14 above:

(40) A There is an island of Atlantis.
B There are inhabitants on the island of Atlantis.
C The inhabitants of the island of Atlantis have blue eyes.

One might think that, since $1_{/C/} = /B/$ and $1_{/B/} = /A/$, it would follow that $1_{1_{/C/}} = /A/$. This would, however, contradict the fact that presuppositional propositions have no presuppositions themselves (see the remark at the end of section 2.3.3), which is stated formally in axiom (D1) of definition 3.3 below: $1_{1_{/C/}} = 1$. What we have in fact is $1_{/C/} = /B/ \cdot /A/ = /B/$ (C presupposes both that there is an island of Atlantis and that it is inhabited, i.e. that the island of Atlantis is inhabited). But our troubles are not over yet: it would follow from $1_{/B/} = /A/$ and $1_{/A/} = 1$ that $1_{/B/} \cdot 1_{/A/} = /A/$, which would lead to the contradiction

$$1 = 1_{1_{/C/}} = 1_{/B/\cdot/A/} = 1_{/B/} \cdot 1_{/A/} = /A/.$$

The mistake in this fallacious argument lies in the fact that $=$ is not a congruence relation. Therefore we cannot replace $1_{/C/}$ with $/A/\cdot/B/$ inside a context (especially under the 1 operator). A counterexample in the formal system PPC_3 is the equality between 1_a and $a + \widetilde{a}$. (In PPC_3, $\widetilde{a}$ is the minimal negation of a.) These two terms, although equal, cannot be substituted for each other in a context. See section 3.4 for a formal treatment.

3.2 The system PPC_3

We now define the formal system of presupposition logic PPC_3. It is an extension of ordinary classical (Boolean) proposition logic with presuppositions and two negations. The propositions are built up from literals, Lit, using the binary connectives $\cdot$ and $+$, the unary connectives 1, $\widetilde{-}$ and $\widehat{-}$ and the constants 0 and 1. The intended meaning of these connectives is this:

1_a the conjoined presuppositions of the sentence a

$\widetilde{a}$ the minimal negation of a (negating a, affirming the presuppositions)

$\widehat{a}$ the radical negation of a (negating the presuppositions)

We use $\overline{-}$ (complement) and $-$ (minus) as *abbreviations* for the composite connectives $\overline{a} := \widetilde{a} + \widehat{a}$ and $a - b := a \cdot \overline{b}$. The intended meaning of $\overline{a}$ is the ordinary Boolean negation, the complement of a. When writing propositions, we remove brackets by letting $\cdot$ bind more strongly than $+$.

We give the precise mathematical definition of the language of PPC_3. Definition 3.1 says that the terms of PPC_3, forming the set T, are constructed starting from the literals, elements of Lit, and the constants 0 and 1, and recursively applying the operators $+$, $\cdot$, $\widehat{-}$, $\widetilde{-}$ and 1.

Definition 3.1 *The set of terms of* PPC_3*,* T*, is defined recursively as follows.*

$$T ::= \mathrm{Lit} \mid T + T \mid T \cdot T \mid 0 \mid 1 \mid \widehat{T} \mid \widetilde{T} \mid 1_T.$$

Remarks 3.2 *1. The connective* $\overline{-}$ *is not taken as a primitive, but is 'decomposed' in terms of other (new) connectives. This means that we have to* prove *that we indeed have a Boolean algebra.*

2. *In Boolean algebra, we can take different sets of connectives as basic (and then define the others in terms of the basic ones). The reason this can be done is that Boolean equality is a congruence with respect to the connectives. In* PPC_3*, equality is not a congruence, hence the choice of primitives is crucial. For example, if we* define $\overline{a} := \widetilde{a} + \widehat{a}$*, as we have done above, we can freely substitute* $\widetilde{a} + \widehat{a}$ *for* $\overline{a}$*, which is not allowed if* $\overline{a} = \widetilde{a} + \widehat{a}$ *is a derived equality. We have already pointed out this problem in section 3.1. A formal analysis is given in section 3.4.*

The Boolean connectives enjoy the well-known Boolean equations. That is, they form a distributive lattice. We recapitulate the axioms of a distributive lattice.

$$
\begin{array}{rcl@{\quad}rcl}
a+b & = & b+a & a\cdot b & = & b\cdot a \\
(a+b)+c & = & a+(b+c) & (a\cdot b)\cdot c & = & a\cdot(b\cdot c) \\
(a+b)\cdot c & = & a\cdot c+b\cdot c & & & \\
a+a & = & a & a\cdot a & = & a \\
a+1 & = & 1 & a\cdot 1 & = & a \\
a+0 & = & a & a\cdot 0 & = & 0
\end{array}
$$

It is well-known that the following equations are now derivable: $a \cdot b + b = b$, $(a + b) \cdot b = b$, $a \cdot b + c = (a + c) \cdot (b + c)$ and $a + b = 0 \to a = 0 \;\&\; b = 0$, $a \cdot b = 1 \to a = 1 \;\&\; b = 1$.

A property which is usually left implicit in the definition of distributive lattice is that $=$ is a congruence for the connectives $\cdot$ and $+$. As '$=$' is *not* a congruence for the other connectives, we need to require this property explicitly by adding the axioms:

$$
\begin{array}{rcl}
a = b \text{ and } c = d & \to & a + c = b + d \\
a = b \text{ and } c = d & \to & a \cdot c = b \cdot d.
\end{array}
$$

Definition 3.3 PPC_3 *is the formal system for deriving equations from*

1. *the axioms for a distributive lattice (including the congruence axioms for* $\cdot$ *and* $+$*, see above),*
2. *the following 10 special axioms*

$$
\begin{array}{lrcl@{\qquad}lrcl}
(A1) & a+\widetilde{a} & = & 1_a & (D1) & 1_{1_a} & = & 1 \\
(A2) & a\cdot\widetilde{a} & = & 0 & (D2) & 1_{\widetilde{a}} & = & 1_a \\
(B1) & \widehat{a}+1_a & = & 1 & (D3) & 1_{\widehat{a}} & = & 1 \\
(B2) & \widehat{a}\cdot 1_a & = & 0 & (D4) & 1_0 & = & 1. \\
(C1) & 1_{a\cdot b} & = & 1_a\cdot 1_b & & & & \\
(C2) & 1_{a+b} & = & 1_a+1_b & & & &
\end{array}
$$

To denote that, for $a, b \in T$*,* $a = b$ *is derivable in* PPC_3*, we shall write*

$$\mathsf{PPC}_3 \vdash a = b$$

Axioms $A1$ and $A2$ state that 1_a is the union of a and $\widetilde{a}$ and that a and $\widetilde{a}$ are disjoint. So, a and $\widetilde{a}$ are each other's complement within 1_a. Axiom $B1$ and $B2$ say something similar about $\widehat{a}$ and 1_a: they are disjoint and their union is 1. This amounts to the first picture in fig. 13, describing $a \subseteq 1_a \subseteq 1$, $\widetilde{a} \subseteq 1_a$ and $\widehat{a} \subseteq 1$ with $a, \widetilde{a}$ disjoint and $1_a, \widehat{a}$ disjoint. Axioms $C1$ and $C2$ specify that the 1 operator commutes with $\cdot$ and $+$. The D-axioms describe how connectives (especially 1, $\widetilde{-}$ and $\widehat{-}$) operate under the 1 connective.

Lemma 3.4 *Given a proposition* a*,* $\overline{a}$ *is the* unique *proposition for which the Boolean laws for complement hold:* $a \cdot \overline{a} = 0$ *and* $a + \overline{a} = 1$*.*

Proof We have to show two things:

1. The defined connective $\overline{-}$ satisfies the axioms of Boolean logic.

2. If $a \cdot b = 0$ and $a + b = 1$, then $b = \overline{a}$ (i.e. $\overline{a}$ is unique).

The proof of the first is as follows.

$$\begin{aligned} a + \overline{a} &= a + \widetilde{a} + \widehat{a} = 1_a + \widehat{a} = 1. \\ a \cdot \overline{a} &= a \cdot (\widetilde{a} + \widehat{a}) = a \cdot \widetilde{a} + a \cdot \widehat{a} = a \cdot \widehat{a} \\ &= a \cdot \widehat{a} + 0 = a \cdot \widehat{a} + 1_a \cdot \widehat{a} = (a + 1_a) \cdot \widehat{a} = (a + a + \widetilde{a}) \cdot \widehat{a} = 1_a \cdot \widehat{a} = 0. \end{aligned}$$

The second is shown as follows. Suppose $a \cdot b = 0$ and $a + b = 1$. Then

$$\overline{a} = \overline{a} \cdot 1 = \overline{a} \cdot (a + b) = \overline{a} \cdot a + \overline{a} \cdot b = \overline{a} \cdot b = a \cdot b + \overline{a} \cdot b = (a + \overline{a}) \cdot b = b.$$

Theorem 3.5 PPC_3 *is an extension of Boolean logic.*

Proof The only thing left to prove is that the equality $=$ is a congruence with respect to the defined connective $\overline{}$, i.e. if $a = b$, then $\overline{a} = \overline{b}$. So, suppose $a = b$. Then $a \cdot \overline{a} = 0$ and $a + \overline{a} = 1$. But, due to the fact that $=$ is a congruence for $\cdot$ and $+$, we also have $b \cdot \overline{a} = 0$ and $b + \overline{a} = 1$. As $\overline{b}$ satisfies these same equations, we conclude that $\overline{a} = \overline{b}$ (by the uniqueness stated in the previous Lemma).

Remark 3.6 *As* PPC_3 *satisfies the Boolean axioms, we can freely use notions from Boolean logic. In the following, we use the abbreviations* $a \leq b$ *(and* $a \geq b$ *for* $b \leq a$*):*

$$\begin{array}{ccc} a \leq b & \textit{abbreviates} & a \cdot b = a \\ & \textit{or, equivalently,} & a + b = b. \end{array}$$

Proposition 3.7 *The following equations are derivable in* PPC_3*.*

$$\begin{array}{lrcl lrcl}
(e1) & 1_a + a & = & 1_a & (e3) & \widehat{a} \cdot a & = & 0 \\
(e2) & 1_a \cdot a & = & a & (e4) & \widetilde{a} \cdot \widehat{a} & = & 0 \\
\\
(e5) & \widetilde{a} & = & 1_a - a & (e11) & \widehat{a} & = & \widetilde{1_a} = \overline{1_a} \\
(e6) & \widehat{\widetilde{a}} & = & 1_a & (e12) & \widehat{\widehat{a}} & = & \widehat{a} \\
(e7) & \widetilde{\widetilde{a}} & = & a & (e13) & \widehat{\widehat{a}} & = & 0 \\
(e8) & \widetilde{a} + \overline{a} & = & \overline{a} & (e14) & \widehat{a} + \overline{a} & = & \overline{a} \\
(e9) & \widetilde{a} \cdot \overline{a} & = & \widetilde{a} & (e15) & \widehat{a} \cdot \overline{a} & = & \widehat{a} \\
(e10) & \widetilde{1_a} & = & \overline{1_a} & (e16) & \widehat{1_a} & = & 0 \\
\\
(e17) & \widetilde{a + b} & = & \widetilde{a} \cdot \widetilde{b} + \widetilde{a} \cdot \widehat{b} + \widehat{a} \cdot \widetilde{b} & (e19) & \widehat{a + b} & = & \widehat{a} \cdot \widehat{b} = \overline{1_{a+b}} \\
(e18) & \widetilde{a \cdot b} & = & \widetilde{a} \cdot \widetilde{b} + \widetilde{a} \cdot b + \widetilde{b} \cdot a & (e20) & \widehat{a \cdot b} & = & \widehat{a} + \widehat{b} = \overline{1_{a \cdot b}} \\
\\
(e21) & 1_1 & = & 1 & & & & \\
(e22) & \widetilde{1} & = & 0 & (e25) & \widetilde{0} & = & 1 \\
(e23) & \widehat{1} & = & 0 & (e26) & \widehat{0} & = & 0 \\
(e24) & \overline{1} & = & 0 & (e27) & \overline{0} & = & 1
\end{array}$$

Proof

(e1) $1_a + a \overset{A1}{=} a + \widetilde{a} + a = \widetilde{a} + a = 1_a$

(e2) $1_a \cdot a \overset{A1}{=} (a + \widetilde{a}) \cdot a = a + 0 = a$

(e3, e4) $0 \overset{B2}{=} 1_a \cdot \widehat{a} \overset{A1}{=} (a + \widetilde{a}) \cdot \widehat{a} = a \cdot \widehat{a} + \widetilde{a} \cdot \widehat{a}$. Hence $a \cdot \widehat{a} = \widetilde{a} \cdot \widehat{a} = 0$.

(e5) $1_a - a = 1_a \cdot (\widetilde{a} + \widehat{a}) = 1_a \cdot \widetilde{a} + 1_a \cdot \widehat{a} \overset{B2}{=} 1_a \cdot \widetilde{a} \overset{A1}{=} (a + \widetilde{a}) \cdot \widetilde{a} \overset{A2}{=} 0 + \widetilde{a} = \widetilde{a}$

$(e10)$ $\widetilde{1_a} \stackrel{e5}{=} 1_{1_a} - 1_a = 1 - 1_a = \overline{1_a}$

$(e11)$ Both $\widehat{a}$ and $\widetilde{1_a}$ are the complement of 1_a(and hence $\widehat{a} = \widetilde{1_a}$) :
$\widehat{a} \cdot 1_a \stackrel{B2}{=} 0$ and $\widehat{a} + 1_a \stackrel{B1}{=} 1$
$\widetilde{1_a} \cdot 1_a \stackrel{A2}{=} 0$ and $\widetilde{1_a} + 1_a \stackrel{A1}{=} 1_{1_a} \stackrel{D1}{=} 1$

$(e12)$ $\widehat{\widetilde{a}} \stackrel{e11}{=} \overline{1_{\widetilde{a}}} \stackrel{D2}{=} \overline{1_a} \stackrel{e11}{=} \widehat{a}$

$(e6)$ $\widetilde{\widehat{a}} \stackrel{e5}{=} 1_{\widehat{a}} - \widehat{a} \stackrel{D3}{=} 1 - \widehat{a} = \overline{\widehat{a}} \stackrel{e11}{=} \overline{\overline{1_a}} = 1_a$

$(e7)$ $\widetilde{\widetilde{a}} = \widetilde{\widetilde{a}} \cdot 1 = \widetilde{\widetilde{a}} \cdot (a + \overline{a}) = \widetilde{\widetilde{a}} \cdot a + \widetilde{\widetilde{a}} \cdot \widetilde{a} + \widetilde{\widetilde{a}} \cdot \widehat{a} \stackrel{A2}{=}$
$\widetilde{\widetilde{a}} \cdot a + 0 + \widetilde{\widetilde{a}} \cdot \widehat{a} \stackrel{e12}{=} \widetilde{\widetilde{a}} \cdot a + \widetilde{\widetilde{a}} \cdot \widehat{\widetilde{a}} \stackrel{e4}{=} \widetilde{\widetilde{a}} \cdot a + 0 = \widetilde{\widetilde{a}} \cdot a + \overline{a} \cdot a =$
$(\widetilde{\widetilde{a}} + \widetilde{a} + \widehat{a}) \cdot a \stackrel{A1}{=} (1_{\widetilde{a}} + \widehat{a}) \cdot a \stackrel{D2}{=} (1_a + \widehat{a}) \cdot a \stackrel{B1}{=} 1 \cdot a = a$

$(e8)$ $\widetilde{a} + \overline{a} = \widetilde{a} + \widetilde{a} + \widehat{a} = \widetilde{a} + \widehat{a} = \overline{a}$

$(e9)$ $\widetilde{a} \cdot \overline{a} = \widetilde{a} \cdot (\widetilde{a} + \widehat{a}) = \widetilde{a} + \widetilde{a} \cdot \widehat{a} = \widetilde{a} \cdot (1 + \widehat{a}) = \widetilde{a}$

$(e13)$ $\widehat{\widehat{a}} \stackrel{e11}{=} \overline{1_{\widehat{a}}} \stackrel{D3}{=} \overline{1} = 0$

$(e14)$ $\widehat{a} + \overline{a} = \widehat{a} + \widetilde{a} + \widehat{a} = \widetilde{a} + \widehat{a} = \overline{a}$

$(e15)$ $\widehat{a} \cdot \overline{a} = \widehat{a} \cdot (\widetilde{a} + \widehat{a}) = \widehat{a} + \widetilde{a} \cdot \widehat{a} = \widehat{a} \cdot (1 + \widetilde{a}) = \widehat{a}$

$(e16)$ $\widehat{1_a} \stackrel{e11}{=} \overline{1_{1_a}} \stackrel{D1}{=} \overline{1} = 0$

$(e17)$ $\widetilde{a + b} \stackrel{e5}{=} 1_{a+b} - (a + b) \stackrel{C2}{=} (1_a + 1_b) - (a + b) = ((1_a - a) - b) + ((1_b - a) - b)) \stackrel{A1}{=}$
$(a + \widetilde{a}) \cdot (\widetilde{a} + \widehat{a}) \cdot (\widetilde{b} + \widehat{b}) + (b + \widetilde{b}) \cdot (\widetilde{a} + \widehat{a}) \cdot (\widetilde{b} + \widehat{b}) \stackrel{A2,e3,e4}{=} \widetilde{a} \cdot \widetilde{b} + \widetilde{a} \cdot \widehat{b} + \widehat{a} \cdot \widetilde{b}$

$(e18)$ $\widetilde{a \cdot b} \stackrel{e5}{=} 1_{a \cdot b} - (a \cdot b) \stackrel{C1}{=} (1_a \cdot 1_b) - (a \cdot b) = (1_a \cdot 1_b) \cdot (\overline{a} + \overline{b}) = (1_a \cdot \overline{a} \cdot 1_b) + (1_b \cdot \overline{b} \cdot 1_a) \stackrel{e5}{=}$
$(\widetilde{a} \cdot 1_b) + (\widetilde{b} \cdot 1_a) \stackrel{A1}{=} (\widetilde{a} \cdot (b + \widetilde{b})) + (\widetilde{b} \cdot (a + \widetilde{a})) = \widetilde{a} \cdot \widetilde{b} + \widetilde{a} \cdot b + \widetilde{b} \cdot a$

$(e19)$ $\widehat{a + b} \stackrel{e11}{=} \overline{1_{a+b}} \stackrel{C2}{=} \overline{1_a + 1_b} = \overline{1_a} \cdot \overline{1_b} \stackrel{e11}{=} \widehat{a} \cdot \widehat{b}$

$(e20)$ $\widehat{a \cdot b} \stackrel{e11}{=} \overline{1_{a \cdot b}} \stackrel{C1}{=} \overline{1_a \cdot 1_b} = \overline{1_a} + \overline{1_b} \stackrel{e11}{=} \widehat{a} + \widehat{b}.$

$(e21)$ $1_1 \stackrel{A1}{=} 1 + \widetilde{1} = 1$

$(e22)$ $\widetilde{1} = \widetilde{1} \cdot 1 \stackrel{A2}{=} 0$

$(e23)$ $\widehat{1} = \widehat{1} \cdot 1 \stackrel{e21}{=} \widehat{1} \cdot 1_1 \stackrel{B2}{=} 0$

$(e24)$ $\overline{1} = \widetilde{1} + \widehat{1} = 0 + 0 = 0$

$(e25)$ $\widetilde{0} = \widetilde{0} + 0 \stackrel{A1}{=} 1_0 \stackrel{D4}{=} 1$

$(e26)$ $\widehat{0} \stackrel{e11}{=} \overline{1_0} \stackrel{D4}{=} \overline{1} \stackrel{e24}{=} 0$

$(e27)$ $\overline{0} = \widetilde{0} + \widehat{0} = 1 + 0 = 1.$

The axioms for PPC_3 given above are still redundant as the connective $\widehat{-}$ is definable in terms of 1 and $\widetilde{-}$.

Lemma 3.8 *In* PPC_3, $\widehat{a}$ *is definable:* $\widehat{a} := \widetilde{1_a}$.

Proof We have to show that, if we remove the connective $\widehat{-}$ and the corresponding axioms, and we define $\widehat{a}$ as above, then all the laws of PPC_3 hold for this defined connective. The only axioms in which $\widehat{-}$ occurs are $B1$, $B2$ and $D3$.

$(B1)$ $\widehat{a} + 1_a = \widetilde{1_a} + 1_a \overset{A1}{=} 1_{1_a} \overset{D1}{=} 1.$

$(B2)$ $\widehat{a} \cdot 1_a = \widetilde{1_a} \cdot 1_a \overset{A2}{=} 0.$

$(D3)$ $1_{\widehat{a}} = 1_{\widetilde{1_a}} \overset{D2}{=} 1_{1_a} \overset{D1}{=} 1.$

So a minimal calculus for PPC_3 would consist of terms (propositions) built up from literals, Lit, using the binary connectives $\cdot$ and $+$, the unary connectives 1 and $\widetilde{-}$ and the constants 0 and 1, satisfying the axioms for a distributive lattice (including congruence axioms for $\cdot$ and $+$), in addition satisfying the axioms $(A1)$, $(A2)$, $(C1)$, $(C2)$, $(D1)$, $(D2)$ and $(D4)$.

3.3 Consistency and models

We can prove consistency of PPC_3 by showing that standard Boolean algebra is a special case of it in which we take $1_a := 1$, $\widehat{a} := 0$ and $\widetilde{a} := \overline{a}$ for every term a. Since Boolean algebra is consistent, PPC_3 must be too. This also implies that the axiom $1_a = 1$ is a consistent extension of PPC_3, yielding the *maximal interpretation* for 1. The parallel *minimal interpretation* $1_a := a$ is not sound, since it conflicts with axiom $D1$: $a = 1_a = 1_{1_a} = 1$, so all propositions would be equal to 1. This shows that no proposition except for the necessarily true ones presupposes itself.

We now define the semantics of PPC_3, inspired by the notions presented in section 2.3.3 and visually displayed in fig. 13. We saw there that to every proposition we can associate two subsets of possible situations, the subset of situations in which the presuppositions are fulfilled, and which we called the subkey of the proposition, and the subset of situations in which the proposition proper is fulfilled. Similarly, we now define two semantic objects associated with a term a of PPC_3, the interpretation of its presupposition, $[\![1_a]\!]$, and the interpretation of a proper, $[\![a]\!]$. We take a more abstract viewpoint than in section 2.3.3, taking as model for the interpretation a general Boolean algebra $\mathsf{B} = \langle B; \sqcap, \sqcup, \bot, \top, \overline{-}\rangle$. Intuitively, think of B as the family of all sets of possible situations, i.e. the powerset of $\mathbf{K}$, of $\bot$ as the empty set $\emptyset$, of $\top$ as the set $\mathbf{K}$ of all situations and of $\sqcap$, $\sqcup$ and $\overline{-}$ as the operations of intersection, union and complementation for sets. We use the relation $\sqsubseteq$, which is defined by: $p \sqsubseteq q$ if $p \sqcap q = p$ or, equivalently, $p \sqcup q = q$. In the case of a set model, $\sqsubseteq$ is the subset relation. We also write $p \sqsupseteq q$ for $q \sqsubseteq p$.

We define a general notion of PPC_3 model. The idea is that to every proposition a we associate two objects, one giving the interpretation of a itself (its Boolean value) and one giving the interpretation of the presuppositions of a (the value of 1_a). An atomic proposition α (a literal) therefore has two basic values, $\rho(\alpha)$ and $\xi(\alpha)$, representing these two interpretations. These basic values are given by two assignments ρ ad ξ, which are parameters of the model. An *assignment* is a map $\rho : \mathrm{Lit} \to B$, from the literals to a Boolean algebra B.

Definition 3.9 *A* PPC_3-model *is a tern* (B, ρ, ξ)*, with* B *a Boolean algebra and* ρ *and* ξ *two assignments such that* $\rho \sqsubseteq \xi$*, i.e.* $\rho(\alpha) \sqsubseteq \xi(\alpha)$ *for every literal* $\alpha \in \mathrm{Lit}$.

Definition 3.10 *Given a* PPC_3-model (B, ρ, ξ)*, the* interpretation function $[\![-]\!]_{\rho\xi}$ *(taking a* PPC_3 *term and returning an element of* B*) is defined as follows.*

$$
\begin{array}{rcl@{\qquad}rcl}
[\![0]\!]_{\rho\xi} & = & \bot, & [\![1_0]\!]_{\rho\xi} & = & \top, \\
[\![1]\!]_{\rho\xi} & = & \top, & [\![1_1]\!]_{\rho\xi} & = & \top, \\
[\![\alpha]\!]_{\rho\xi} & = & \rho(\alpha), & [\![1_\alpha]\!]_{\rho\xi} & = & \xi(\alpha), \\
[\![a \cdot b]\!]_{\rho\xi} & = & [\![a]\!]_{\rho\xi} \sqcap [\![b]\!]_{\rho\xi}, & [\![1_{a\cdot b}]\!]_{\rho\xi} & = & [\![1_a]\!]_{\rho\xi} \sqcap [\![1_b]\!]_{\rho\xi}, \\
[\![a + b]\!]_{\rho\xi} & = & [\![a]\!]_{\rho\xi} \sqcup [\![b]\!]_{\rho\xi}, & [\![1_{a+b}]\!]_{\rho\xi} & = & [\![1_a]\!]_{\rho\xi} \sqcup [\![1_b]\!]_{\rho\xi}, \\
[\![\widetilde{a}]\!]_{\rho\xi} & = & [\![1_a]\!]_{\rho\xi} - [\![a]\!]_{\rho\xi}, & [\![1_{\widetilde{a}}]\!]_{\rho\xi} & = & [\![1_a]\!]_{\rho\xi}, \\
[\![\widehat{a}]\!]_{\rho\xi} & = & \overline{[\![1_a]\!]_{\rho\xi}}, & [\![1_{\widehat{a}}]\!]_{\rho\xi} & = & \top, \\
[\![1_{1_a}]\!]_{\rho\xi} & = & \top. & & &
\end{array}
$$

Remark 3.11 *Note that the interpretation function* $[\![-]\!]$ *is well-defined, but not by induction on the* length *of a proposition, but by induction on the measure* m*, defined as follows.* $m(\alpha) = 1$, $m(a + b) = m(a) + m(b)$*,* $m(a \cdot b) = m(a) + m(b)$*,* $m(1_a) = 1 + m(a)$*,* $m(\widetilde{a}) = m(\widehat{a}) = 2 + m(a)$*.*

The property that the interpretation of a proposition is always contained in the interpretation of its presuppositions is expressed by the following lemma.

Lemma 3.12 *In a* PPC$_3$*-model we have*

$$[\![1_a]\!]_{\rho\xi} \sqsupseteq [\![a]\!]_{\rho\xi}.$$

Proof Remembering that in a Boolean algebra $b_1 \sqsupseteq b_2$ is defined as $b_1 \sqcap b_2 = b_2$ or, equivalently, as $b_1 \sqcup b_2 = b_1$, we prove the claim by induction on the structure of a.

$$\begin{array}{rl}
\alpha & [\![1_\alpha]\!]_{\rho\xi} \sqcap [\![\alpha]\!]_{\rho\xi} = \xi(\alpha) \sqcap \rho(\alpha) = \rho(\alpha) \quad \text{(because } \rho \sqsubseteq \xi\text{)} \\
\widetilde{a} & [\![1_{\widetilde{a}}]\!]_{\rho\xi} \sqcap [\![\widetilde{a}]\!]_{\rho\xi} = [\![1_a]\!]_{\rho\xi} \sqcap [\![1_a]\!]_{\rho\xi} \sqcap \overline{[\![a]\!]_{\rho\xi}} = [\![1_a]\!]_{\rho\xi} \sqcap \overline{[\![a]\!]_{\rho\xi}} = [\![\widetilde{a}]\!]_{\rho\xi} \\
\widehat{a} & [\![1_{\widehat{a}}]\!]_{\rho\xi} \sqcap [\![\widehat{a}]\!]_{\rho\xi} = \top \sqcap [\![\widehat{a}]\!]_{\rho\xi} = [\![\widehat{a}]\!]_{\rho\xi} \\
1_a & [\![1_{1_a}]\!]_{\rho\xi} \sqcap [\![1_a]\!]_{\rho\xi} = \top \sqcap [\![1_a]\!]_{\rho\xi} = [\![1_a]\!]_{\rho\xi} \\
a \cdot b & [\![1_{a\cdot b}]\!]_{\rho\xi} \sqcap [\![a \cdot b]\!]_{\rho\xi} = [\![1_a]\!]_{\rho\xi} \sqcap [\![a]\!]_{\rho\xi} \sqcap [\![1_b]\!]_{\rho\xi} \sqcap [\![b]\!]_{\rho\xi} \overset{IH}{=} [\![a]\!]_{\rho\xi} \sqcap [\![b]\!]_{\rho\xi} = [\![a \cdot b]\!]_{\rho\xi} \\
a + b & [\![1_{a+b}]\!]_{\rho\xi} \sqcup [\![a + b]\!]_{\rho\xi} = [\![1_a]\!]_{\rho\xi} \sqcup [\![a]\!]_{\rho\xi} \sqcup [\![1_b]\!]_{\rho\xi} \sqcup [\![b]\!]_{\rho\xi} \overset{IH}{=} [\![1_a]\!]_{\rho\xi} \sqcup [\![1_b]\!]_{\rho\xi} = [\![1_{a+b}]\!]_{\rho\xi}
\end{array}$$

where $\overset{IH}{=}$ denotes an application of the *induction hypothesis*, stating that the thesis already holds for a and b. Note that only in the last case do we use (for convenience) $b_1 \sqcup b_2 = b_1$ as a formulation for $b_1 \sqsupseteq b_2$.

The two main properties that we expect from a semantics are *validity* and *completeness*. Validity states that every equality $a = b$ that can be proved in the system is *valid*, i.e. the interpretations of the two terms, $[\![a]\!]_{\rho\xi}$ and $[\![b]\!]_{\rho\xi}$, are the same in every model. This guarantees that what we derive formally is true. Completeness states that if two terms a and b are interpreted in equal objects in every model, then it must be possible to prove that they are equal, i.e. PPC$_3 \vdash a = b$ is derivable. This guarantees that our formal system completely captures all the properties of the semantics.

Theorem 3.13 (Validity) *The model notion of Definition 3.9 is* sound*, i.e. if* PPC$_3 \vdash a = b$*, then* $[\![a]\!]_{\rho\xi} = [\![b]\!]_{\rho\xi}$ *in all* PPC$_3$*-models* (B, ρ, ξ).

Proof We have to check that the axioms for a distributive lattice and the 10 axioms of Definition 3.3 hold in the model.

The axioms for a distributive lattice are trivially proved from the fact that B is a distributive lattice.

That axioms $(C1)$, $(C2)$, $(D1)$–$(D4)$ hold in the model follows immediately from the definition of the interpretation (3.10). Rules $(A1)$–$(B2)$ require slightly more work. We show $(A2)$, $(B1)$ and $(B2)$ in detail and then we discuss $(A1)$.

$$\begin{array}{ll}
(A2) & [\![a \cdot \widetilde{a}]\!]_{\rho\xi} = [\![a]\!]_{\rho\xi} \sqcap ([\![1_a]\!]_{\rho\xi} - [\![a]\!]_{\rho\xi}) = [\![a]\!]_{\rho\xi} \sqcap [\![1_a]\!]_{\rho\xi} \sqcap \overline{[\![a]\!]_{\rho\xi}} = \bot \\
(B1) & [\![\widehat{a} + 1_a]\!]_{\rho\xi} = \overline{[\![1_a]\!]_{\rho\xi}} \sqcup [\![1_a]\!]_{\rho\xi} = \top \\
(B2) & [\![\widehat{a} \cdot 1_a]\!]_{\rho\xi} = \overline{[\![1_a]\!]_{\rho\xi}} \sqcap [\![1_a]\!]_{\rho\xi} = \bot
\end{array}$$

To prove that (A1) holds, we first recall that $[\![1_a]\!]_{\rho\xi} \sqsupseteq [\![a]\!]_{\rho\xi}$ for every a, or equivalently, that $[\![1_a]\!]_{\rho\xi} \sqcap [\![a]\!]_{\rho\xi} = [\![a]\!]_{\rho\xi}$ for every a. This was proved in Lemma 3.12. Given this result, we prove $(A1)$ as follows.

$$\begin{array}{ll}
(A1) & [\![a + \widetilde{a}]\!]_{\rho\xi} = [\![a]\!]_{\rho\xi} \sqcup ([\![1_a]\!]_{\rho\xi} \sqcap \overline{[\![a]\!]_{\rho\xi}}) = ([\![1_a]\!]_{\rho\xi} \sqcap [\![a]\!]_{\rho\xi}) \sqcup ([\![1_a]\!]_{\rho\xi} \sqcap \overline{[\![a]\!]_{\rho\xi}}) = \\
& [\![1_a]\!]_{\rho\xi} \sqcap ([\![a]\!]_{\rho\xi} \sqcup \overline{[\![a]\!]_{\rho\xi}}) = [\![1_a]\!]_{\rho\xi}
\end{array}$$

To prove completeness we define the PPC$_3$-*term-model*. This is a PPC$_3$-model consisting of the terms of PPC$_3$ (given by the set T, see Definition 3.1) itself. This means that we have to cast T into a Boolean algebra and define ρ and ξ as required by Definition 3.9.

Definition 3.14 *The set $\mathcal{B}$ is defined by quotienting T with the* PPC_3*-equality. In other words, the elements of $\mathcal{B}$ are the* equivalence classes $[t]$ *(for $t \in T$), where*

$$[t] := \{t' \in T \mid \mathsf{PPC}_3 \ \vdash t = t'\}.$$

The Boolean operations are defined as the corresponding operators of PPC_3 *applied inside the equivalence classes:*

$$\begin{array}{rcl} \bot & := & [0], \\ [a] \sqcap [b] & := & [a \cdot b], \\ \overline{[a]} & := & [\overline{a}] = [\widehat{a} + \widetilde{a}]. \end{array} \qquad \begin{array}{rcl} \top & := & [1], \\ [a] \sqcup [b] & := & [a + b], \end{array}$$

It can be proved that these operations are well-defined and they determine a Boolean algebra.

The PPC_3-term-model *is now obtained by taking $(\mathcal{B}, \rho, \xi)$ with $\rho(\alpha) = [\alpha]$ and $\xi(\alpha) = [1_\alpha]$ for $\alpha \in \mathrm{Lit}$.*

Lemma 3.15 *1. The* PPC_3*-term-model ($(\mathcal{B}, \rho, \xi)$ in the previous Definition) is indeed a* PPC_3*-model.*

2. For all $a, b \in T$, if $[\![a]\!]_{\rho\xi} = [\![b]\!]_{\rho\xi}$ in the PPC_3*-term-model $(\mathcal{B}, \rho, \xi)$, then $\mathsf{PPC}_3 \vdash a = b$.*

Proof 1. It has to be shown that $\mathcal{B}$ is a Boolean algebra and that $\rho \sqsubseteq \xi$. The first follows from Theorem 3.5. The second follows from the fact that $1_a \cdot a = a$ is a derived rule in PPC_3 (rule (e2) in Proposition 3.7).

2. This follows immediately from the fact that

$$[\![a]\!]_{\rho\xi} = [a]$$

for all $a \in T$, which can be shown by an easy induction on the structure of a.

Theorem 3.16 (Completeness) *The model notion of Definition 3.9 is* complete*, i.e. if $[\![a]\!]_{\rho\xi} = [\![b]\!]_{\rho\xi}$ holds in all* PPC_3*-models (B, ρ, ξ), then $\mathsf{PPC}_3 \vdash a = b$.*

Proof Suppose a and b are two PPC_3-terms such that

$$[\![a]\!]_{\rho\xi} = [\![b]\!]_{\rho\xi}$$

holds in all PPC_3-models. Then $[\![a]\!]_{\rho\xi} = [\![b]\!]_{\rho\xi}$ holds in the PPC_3-term-model $(\mathcal{B}, \rho, \xi)$ and hence $\mathsf{PPC}_3 \vdash a = b$, due to Lemma 3.15.

Corresponding to the maximal interpretation of PPC_3, we have trivial models in which $\xi(\alpha) = \top$ for every $\alpha \in \mathrm{Lit}$. We can construct simple nontrivial models by choosing any boolean algebra B and any function ρ that is not constantly $\top$ and letting $\xi(\alpha) = \rho(\alpha)$ for every $\alpha \in \mathrm{Lit}$. In this model we have that for a literal α such that $\rho(\alpha) \neq \top$, also $[\![1_\alpha]\!]_{\rho\xi} = \xi(\alpha) \neq \top = [\![1]\!]_{\rho\xi}$ and so the model is nontrivial (the interpretation of 1_A is not just always $\top$). Observe that this model does not correspond to the unsound minimal interpretation, because the identification of the presupposition of a term with the term itself is stipulated only for the literals and not for every term. Notably, for a term 1_α (α a literal), the presupposition of 1_α is not identified with 1_α in the model. (Proof: we have seen that $[\![1_\alpha]\!]_{\rho\xi} \neq \top$. It is also the case that $[\![1_{1_\alpha}]\!]_{\rho\xi} = \top$, so $[\![1_\alpha]\!]_{\rho\xi} \neq [\![1_{1_\alpha}]\!]_{\rho\xi}$: the terms 1_α and 1_{1_α} are not identified in this model.) It also follows from validity that $1_\alpha = 1$ is not derivable in PPC_3. As a consequence, the term model is also nontrivial. Indeed, for a literal α, we have that $[\![1_\alpha]\!]_{\rho\xi} \neq [\![1]\!]_{\rho\xi}$, because the equality $1_\alpha = 1$ is not derivable in PPC_3. As a conclusion of this paragraph we state the following fact.

Fact 3.17 *There are non-trivial models of* PPC_3*, that is, models in which $[\![1_a]\!]_{\rho\xi} \neq [\![1]\!]_{\rho\xi}$ for some term a.*

3.4 Compositionality in the calculus and the models

In section 3.1 the noncompositionality of the calculus has already been discussed when we looked at the example sentences (40). It has been argued that the equality of PPC_3 is not a congruence. A counterexample to congruence in the formal system PPC_3 is given by the terms $a+\widetilde{a}$ and 1_a, which are equal but cannot be substituted for each other in a context. According to axiom $(A1)$, $1_a = a+\widetilde{a}$. If the calculus were compositional, we could substitute one of the two terms for the other when they appear as arguments of the 1 operator, yielding $1_{1_a} = 1_{a+\widetilde{a}}$. But this equality is not always valid. In fact, the left hand side is equal to the unity, $1_{1_a} = 1$ by $(D1)$, whereas the right hand side can be proved to be equal to the presupposition of a, $1_{a+\widetilde{a}} = 1_a + 1_{\widetilde{a}} = 1_a + 1_a = 1_a$. Unless the presupposition of a is trivially equal to unity, the supposed equality cannot hold. This clearly shows that in PPC_3 equality is not a congruence with respect to the 1 operator.

Similarly, equality is not a congruence with respect the operators $\widetilde{-}$ and $\widehat{-}$. A counterexample is again given by the term $a+\widetilde{a}$: $1_a = a+\widetilde{a}$, but $\widetilde{1_a} = \widetilde{a+\widetilde{a}}$ is not generally true, because $\widetilde{a+\widetilde{a}} = \widetilde{a}\cdot\widetilde{\widetilde{a}} + \widetilde{a}\cdot\widehat{\widetilde{a}} + \widehat{a}\cdot\widetilde{\widetilde{a}} = 0 + \widetilde{a}\cdot\widehat{a} + \widehat{a}\cdot a = 0+0 = 0$ (using $e17$, $e12$ and $e7$) and $\widetilde{1_a} = \widehat{a}$. Finally $1_a = a+\widetilde{a}$, but $\widehat{1_a} = \widehat{a+\widetilde{a}}$ is not true in general, because $\widehat{a+\widetilde{a}} = \widehat{a}\cdot\widehat{\widetilde{a}} = \widehat{a}$ (using $e19$ and $e12$) and $\widehat{1_a} = 0$.

In a model, the fact that equality is a congruence with respect to the operators is called *compositionality*: the interpretation of a formula is defined by structural recursion. For the 1 operator, this would mean that $[1_a]$ is defined as $U([a])$, with U the function that represents the 1 operator in the model. In a non-trivial model, we cannot have such an operator U. Or, stated differently, the operator U in the model (that represents the connective 1) cannot be a function.

Definition 3.18 *We call* compositionality *the rule*

$$a = b \to 1_a = 1_b.$$

Fact 3.19 *Compositionality is equivalent to the rule* $1_a = 1$.

Proof Assume compositionality. Then $1_a = 1_{\widetilde{a}} = 1_{1_a\cdot\overline{a}} = 1_{1_a}\cdot 1_{\overline{a}} = 1\cdot 1_{\widetilde{a}+\widehat{a}} = 1_{\widetilde{a}} + 1_{\widehat{a}} = 1_a + 1 = 1$. Proving compositionality from $1_a = 1$ is easy.

So, compositionality yields a trivial model. We have a way of constructing non-trivial models: If we take $\xi(\alpha) \neq \top$, then $\llbracket 1_\alpha \rrbracket_{\rho\xi} \neq \top$. In such a model compositionality does not hold: 1_α and $\alpha+\widetilde{\alpha}$ are equal in a non-trivial model, but 1_{1_α} and $1_{\alpha+\widetilde{\alpha}}$ are not: $\llbracket 1_{1_\alpha} \rrbracket_{\rho\xi} = \top$, whereas $\llbracket 1_{\alpha+\widetilde{\alpha}} \rrbracket_{\rho\xi} = \xi(\alpha)$.

Remarks 3.20
1. *If we let $\xi(\alpha) = \top$ for all literals, we have a trivial model (i.e. $\llbracket 1_a \rrbracket_{\rho\xi} = \top$ for all a).*
2. *There can be no model in which $\llbracket a \rrbracket_{\rho\xi} = \llbracket 1_a \rrbracket_{\rho\xi}$ for all a. Suppose that $\llbracket a \rrbracket_{\rho\xi} = \llbracket 1_a \rrbracket_{\rho\xi}$ for all a. Then $\llbracket a \rrbracket_{\rho\xi} = \llbracket 1_a \rrbracket_{\rho\xi} = \llbracket 1_{1_a} \rrbracket_{\rho\xi} = \top$ for all a. This is a contradiction because at least 0 is not interpreted as $\top$.*

3.5 A compositional equality in PPC_3

We have already observed that in PPC_3 there is only one 'level' of presuppositions: if a is a sentence, then 1_a, the sentence that expresses the presuppositions of a, is, in general, a sentence different from 1. But 1_{1_a}, the sentence expressing the presuppositions of 1_a (the presuppositions of the presuppositions of a) is always 1. So, two sentences a and b can be distinct in their 'classical' Boolean interpretation (then $a \neq b$) or they can be distinct in their presuppositions (then $(1_a \neq 1_b)$, but in no other way: we always have $1_{1_a} = 1_{1_b}$. This fact can also be observed in a different way. We first define the *strong equality* $a \equiv b$.

Definition 3.21 *The* strong equality $a \equiv b$ *in* PPC_3 *is defined as follows.*

$$a \equiv b \text{ if and only if } a = b \text{ and } 1_a = 1_b \text{ in } \mathsf{PPC}_3.$$

Lemma 3.22 *Strong equality is a congruence for all connectives. That is*

$$\begin{aligned}
a \equiv b \,\&\, c \equiv d &\rightarrow a + c \equiv b + d && (1)\\
a \equiv b \,\&\, c \equiv d &\rightarrow a \cdot c \equiv b \cdot d && (2)\\
a \equiv b &\rightarrow 1_a \equiv 1_b && (3)\\
a \equiv b &\rightarrow \widetilde{a} \equiv \widetilde{b} && (4)\\
a \equiv b &\rightarrow \widehat{a} \equiv \widehat{b} && (5)\\
a \equiv b &\rightarrow \overline{a} \equiv \overline{b}. && (6)
\end{aligned}$$

Proof Suppose $a \equiv b$ and $c \equiv d$. Then $a = b$, $1_a = 1_b$, $c = d$ and $1_c = 1_d$. Hence $a + c = b + d$, because $=$ is a congruence for $+$. We also find

$$1_{a+c} \stackrel{C1}{=} 1_a + 1_c = 1_b + 1_d \stackrel{C1}{=} 1_{b+d},$$

and so $a + c \equiv b + d$. The argument for $\cdot$ is analogous. Therefore we have proved (1) and (2).
Suppose $a \equiv b$. Then $a = b$ and $1_a = 1_b$. As $1_{1_a} = 1_{1_b}$ by rule $(D1)$, we find that $1_a \equiv 1_b$, which proves (3).
Suppose $a \equiv b$. Then $a = b$ and $1_a = 1_b$. Now, $\widetilde{a} \stackrel{e5}{=} 1_a \cdot \overline{a} = 1_b \cdot \overline{b} \stackrel{e5}{=} \widetilde{b}$. Also $1_{\widetilde{a}} \stackrel{D2}{=} 1_a = 1_b \stackrel{D2}{=} 1_{\widetilde{b}}$, which proves (4).
Suppose $a \equiv b$. Then $a = b$ and $1_a = 1_b$. Now, $\widehat{a} \stackrel{e11}{=} \overline{1_a} = \overline{1_b} \stackrel{e11}{=} \widehat{b}$. Also $1_{\widehat{a}} \stackrel{D3}{=} 1 \stackrel{D3}{=} 1_{\widehat{b}}$, thus proving (5).
Suppose $a \equiv b$. Then $a = b$ and $1_a = 1_b$. Using (4) and (5) we find that $\overline{a} = \widehat{a} + \widetilde{a} = \widehat{b} + \widetilde{b} = \overline{b}$. Using (3), (4) and (5), we also derive that $1_{\overline{a}} = 1_{\widetilde{a}+\widehat{a}} \stackrel{C2}{=} 1_{\widetilde{a}} + 1_{\widehat{a}} = 1_{\widetilde{b}} + 1_{\widehat{b}} \stackrel{C2}{=} 1_{\widetilde{b}+\widehat{b}} = 1_{\overline{b}}$, thus proving (6).

3.6 A compositional presentation of PPC_3

Building on the previous section, we give a completely compositional presentation of PPC_3. That is, we characterize the compositional equality $\equiv$ independently. Moreover, we define the (non-compositional) equality of PPC_3 in terms of this $\equiv$. We call our new system PPC_3^c, compositional PPC_3.

There are two reasons for studying this new system. First, our aim in developing a formal system for presuppositional sentences is to capture the logic and semantics of presuppositions. The meaning of a proposition contains the meaning of its presuppositions. It is natural to say that two propositions are equal when they have the same meaning. Since there is no precise mathematical theory of meaning, this cannot be done in a Boolean setting, in which the equality "=" is taken to be identity of extensions. Much of the meaning of a proposition is lost in this interpretation. We have made an effort to produce a mathematical theory that captures a little more of the meaning of sentences. We are now in the position to give an interpretation of propositions which is more faithful to what really happens in natural language. Hence, we consider two propositions to be equal when not just their extensions, but also the extensions of their presuppositions coincide.

Second, a compositional theory has nicer mathematical properties that facilitate its study. PPC_3^c is a standard equational theory, that can be studied using classical methods from Universal Algebra. Once the equivalence of PPC_3 and PPC_3^c is established, it is easier, when trying to prove something in PPC_3, to translate the problem into the system PPC_3^c and solve it there.

Definition 3.23 *The language of PPC_3^c is almost the same as that of PPC_3. There are two additions. The first is the constant $\odot$, which indicates a proposition that presupposes a necessary falsity. The second is a new unary operation $\blacktriangle$ that takes a proposition as argument and yields a necessarily false proposition having the argument as presupposition.*

The set of terms of PPC_3^c, T^c, is defined recursively as follows.

$$T^c ::= \mathrm{Lit} \,|\, T^c + T^c \,|\, T^c \cdot T^c \,|\, \odot \,|\, 1 \,|\, \widetilde{T^c} \,|\, 1_{T^c} \,|\, \blacktriangle_{T^c}.$$

Definition 3.24 *The axioms of* PPC_3^c *are the following.*

1. *The equality* $\equiv$ *is a congruence relation, i.e. is an equivalence relation and is preserved under application of the operations: if* $a_1 \equiv a_2$ *and* $b_1 \equiv b_2$ *then* $a_1 + b_1 \equiv a_2 + b_2$, $a_1 \cdot b_1 \equiv a_2 \cdot b_2$, $\widetilde{a_1} \equiv \widetilde{a_2}$, $1_{a_1} \equiv 1_{a_2}$ *and* $\blacktriangle_{a_1} \equiv \blacktriangle_{a_2}$.

2. *The operators* $+$ *and* $\cdot$ *and the constants* $\odot$ *and* 1 *determine a distributive lattice with a bottom and a top element. This means that the following equations hold.*

$$
\begin{array}{cc}
a + b \equiv b + a & a \cdot b \equiv b \cdot a \\
a + (b + c) \equiv (a + b) + c & a \cdot (b \cdot c) \equiv (a \cdot b) \cdot c \\
(a \cdot b) + b \equiv b & (a + b) \cdot b \equiv b \\
(a + b) \cdot c \equiv a \cdot c + b \cdot c & (a \cdot b) + c \equiv (a + c) \cdot (b + c) \\
a + a \equiv a & a \cdot a \equiv a \\
a + 1 \equiv 1 & a \cdot \odot \equiv \odot \\
a + \odot \equiv a & a \cdot 1 \equiv a
\end{array}
$$

3. *Specific axioms for* PPC_3^c *that determine the properties of the unary operators* $\tilde{-}$, 1 *and* $\blacktriangle$.

$$
\begin{array}{lrcllrcl}
(A_1^c) & \widetilde{1_\odot} & \equiv & 1 & (A_7^c) & 1_{1_a} & \equiv & 1 \\
(A_2^c) & \widetilde{\widetilde{a}} & \equiv & a & (A_8^c) & 1_{\widetilde{a}} & \equiv & 1_a \\
(A_3^c) & a \cdot \widetilde{a} & \equiv & a \cdot 1_\odot & (A_9^c) & 1_a \cdot a & \equiv & a \\
(A_4^c) & a + \widetilde{a} + 1_\odot & \equiv & 1_a & (A_{10}^c) & 1_\odot + \widetilde{1}_a & \equiv & \widetilde{1}_a \\
(A_5^c) & 1_{a \cdot b} & \equiv & 1_a \cdot 1_b & (A_{11}^c) & 1_{\blacktriangle_a} & \equiv & a + 1_\odot \\
(A_6^c) & 1_{a+b} & \equiv & 1_a + 1_b & (A_{12}^c) & a & \equiv & \widetilde{\blacktriangle_{1_a}} \cdot 1_{\blacktriangle_a}
\end{array}
$$

Note 3.25 *The symbol* $\odot$ *is not the usual zero, it does not correspond to* 0 *in the original presentation of* PPC_3. *It is rather an* absolute zero *corresponding to propositions that presuppose a necessary falsity, like for example*

John knows that bachelors are married.

The operator $\blacktriangle$, *when applied to a proposition* a, *gives a proposition* $\blacktriangle_a$ *which is necessary false and has* a *as presupposition. An example of such a construction in language could be*

Some living dead know that a.

Note 3.26 *We do not require that our structure is a Boolean algebra. Indeed the negation operation* $\tilde{-}$ *does not behave like the ordinary complement in Boolean algebras. Specifically the equation* $a \cdot \widetilde{a} \equiv \odot$ *is not satisfied.*

We want to prove that this theory is equivalent to the original one. We first define the missing symbols.

Definition 3.27

$$
\begin{array}{rcl}
0 & := & 1_\odot \\
\widehat{a} & := & \widetilde{1}_a \\
\overline{a} & := & \widehat{a} + \widetilde{a} \\
a - b & := & a \cdot \overline{b}
\end{array}
$$

and the (weak) equality

$$a = b \overset{\text{def}}{\Longleftrightarrow} a + 0 \equiv b + 0.$$

Now we have to prove that with these definitions $\cdot$, $+$, 0, 1, $\overline{-}$ and $=$ form a Boolean algebra and that the axioms of 3.3 are satisfied.

Lemma 3.28 $\cdot$, $+$ *and* $=$ *form a distributive lattice.*

Proof It is enough to sum 0 to both sides of the corresponding equations that express the fact that $\cdot$, $+$ and $\equiv$ form a distributive lattice. Of the two axioms involving 0, the first, $a+0=a$, translates to $a+1_\odot+1_\odot \equiv 1_\odot$, which is trivially true. The second, $a\cdot 0 = 0$ translates to $a\cdot 1_\odot + 1_\odot \equiv 1_\odot$ and is proved by the following argument:

$$a\cdot 1_\odot + 1_\odot \equiv a\cdot 1_\odot + 1\cdot 1_\odot \equiv (a+1)\cdot 1_\odot \equiv 1\cdot 1_\odot \equiv 1_\odot.$$

Lemma 3.29 $a\cdot\overline{a} = 0$

Proof If we unfold the definitions we have to prove that $a\cdot(\widetilde{1_a}+\widetilde{a})+1_\odot \equiv 1_\odot$.

$$\begin{array}{rclcl}
a\cdot(\widetilde{1_a}+\widetilde{a})+1_\odot & \equiv & a\cdot\widetilde{1_a}+a\cdot\widetilde{a}+1_\odot & \overset{A_3^c}{\equiv} & a\cdot\widetilde{1_a}+a\cdot 1_\odot+1_\odot \\
& \equiv & a\cdot(\widetilde{1_a}+1_\odot)+1_\odot & \overset{A_{10}^c}{\equiv} & a\cdot\widetilde{1_a}+1_\odot \\
& \overset{A_9^c}{\equiv} & a\cdot 1_a\cdot\widetilde{1_a}+1_\odot & \overset{A_3^c}{\equiv} & a\cdot 1_a\cdot 1_\odot+1_\odot \\
& \overset{A_9^c}{\equiv} & a\cdot 1_\odot+1_\odot & \equiv & (a+1)\cdot 1_\odot \\
& \equiv & 1\cdot 1_\odot & \equiv & 1_\odot.
\end{array}$$

Lemma 3.30 $a+\overline{a} = 1$.

Proof Unfolding the definitions, we have to prove that $a+\widetilde{1_a}+\widetilde{a}+1_\odot \equiv 1$.

$$\begin{array}{rclcl}
a+\widetilde{1_a}+\widetilde{a}+1_\odot & \equiv & a+\widetilde{a}+1_\odot+\widetilde{1_a}+1_\odot & \overset{A_4^c}{\equiv} & 1_a+\widetilde{1_a}+1_\odot \\
& \overset{A_4^c}{\equiv} & 1_{1_a} & \overset{A_7^c}{\equiv} & 1.
\end{array}$$

We have thus proved that

Theorem 3.31 $\cdot$, $+$, 1, 0, $\overline{}$ *and* $=$ *determine a Boolean algebra.*

We prove the specific equalities of PPC_3.

Proposition 3.32 *The axioms A1–D4 of* PPC_3 *are satisfied in* PPC_3^c.

Proof *A*1 $a+\widetilde{a} = 1_a$. Immediate from A_4^c.

*A*2 $a\cdot\widetilde{a} = 0$. We have to prove that $a\cdot\widetilde{a}+0 \equiv 0+0$, i.e. $a\cdot\widetilde{a}+0\equiv 0$.
$a\cdot\widetilde{a}+0 \overset{A_3^c}{\equiv} a\cdot 1_\odot + 0 \equiv a\cdot 0+0 \equiv (a+1)\cdot 0 \equiv 1\cdot 0 = 0$.

*B*1 $\widehat{a}+1_a = 1$. Unfolding some of the definitions we have to prove that $\widetilde{1_a}+1_a+1_\odot \equiv 1+0$.
Now $\widetilde{1_a}+1_a+0 \overset{A_4^c}{\equiv} 1_{1_a} \overset{A_7^c}{\equiv} 1 = 1+0$.

*B*2 $\widehat{a}\cdot 1_a = 0$. Unfolding some of the definitions we have to prove that $\widetilde{1_a}\cdot 1_a + 0 \equiv 1+0$. Now $\widetilde{1_a}\cdot 1_a \overset{A_3^c}{\equiv} 1_a\cdot 1_\odot \overset{A_5^c}{\equiv} 1_{a\cdot\odot} \equiv 1_\odot \equiv 0$, proving the claim.

*C*1 $1_{a\cdot b} = 1_a\cdot 1_b$. Immediate from A_5^c.

*C*2 $1_{a+b} = 1_a + 1_b$. Immediate from A_6^c.

*D*1 $1_{1_a} = 1$. Immediate from A_7^c.

*D*2 $1_a = 1_{\widetilde{a}}$. Immediate from A_8^c.

*D*3 $1_{\widehat{a}} = 1$. Unfolding the definitions we have to prove that $1_{\widetilde{1_a}}+0 \equiv 1+0$. Now the claim follows from $1_{\widetilde{1_a}} \overset{A_8^c}{\equiv} 1_{1_a} \overset{A_7^c}{\equiv} 1$.

*D*4 $1_0 = 1$. Unfolding the definitions we have to prove that $1_{1_\odot} + 0 \equiv 1 + 0$. This follows immediately from (A_7^c).

So the axioms of the original PPC_3 are satisfied.

Theorem 3.33 PPC_3^c *with the defined weak equality satisfies the axioms of* PPC_3*.*

Vice versa, if we start with the original PPC_3 and we define

$$a \equiv b \overset{\text{def}}{\iff} a = b \text{ and } 1_a = 1_b$$

we can prove that the axioms of PPC_3^c are satisfied, provided that we give the following definition for the extra symbols.

$$\odot := 0 \qquad \blacktriangle_a := 0$$
$$1_\odot := 0 \qquad 1_{\blacktriangle_a} := a$$

Note that in these definitions we must specify not only the value of the defined term but also that of its presupposition, owing to the noncompositionality of the system. Since these definitions extend the domain of the operator 1, we must check that the axioms pertaining to it are still satisfied.

Theorem 3.34 PPC_3 *with the defined strong equality satisfies the axioms of* PPC_3^c*.*

But these embedding theorems are still too weak. Suppose we start out with the system PPC_3^c with the strong equality $\equiv$. We now define the weak equality $=$ as

$$a = b \overset{\text{def}}{\iff} a + 0 \equiv b + 0.$$

We know that this equality satisfies the axioms of PPC_3. From this equality we now define a new strong equality by

$$a \equiv' b \overset{\text{def}}{\iff} a = b \text{ and } 1_a = 1_b$$

We now want to prove that this strong equality coincides with the original one.

Lemma 3.35 $1_a + 0 \equiv 1_a$

Proof Easy.

Theorem 3.36 $a \equiv b$ *if and only if* $a \equiv' b$*.*

Proof From left to right, $a \equiv b \to a \equiv' b$, is immediate by substitution.

From right to left, $a \equiv' b \to a \equiv b$, needs some reasoning. Assume that $a \equiv' b$ holds. If we unfold the definition of $\equiv'$ we obtain that $a = b$ and $1_a = 1_b$. If we unfold also the definition of $=$ we obtain that $a + 0 \equiv b + 0$ and $1_a + 0 \equiv 1_b + 0$. From these equalities we want to derive that $a \equiv b$. From the first equality and axiom A_{11}^c we have that $1_{\blacktriangle_a} \equiv 1_{\blacktriangle_b}$. From the second equality and lemma 3.35 we have that $1_a \equiv 1_b$. Now by axiom A_{12}^c we have that

$$a \equiv \widetilde{\blacktriangle_{1_a}} \cdot 1_{\blacktriangle_a} \equiv \widetilde{\blacktriangle_{1_b}} \cdot 1_{\blacktriangle_b} \equiv b$$

as desired.

An interesting property is the following.

Lemma 3.37 $\widetilde{a+b} \equiv \widetilde{a} \cdot \overline{b} + \overline{a} \cdot \widetilde{b}$.

Proof The proof is given in the original system PPC_3, i.e. we prove that $\widetilde{a+b} = \widetilde{a}\cdot\overline{b} + \overline{a}\cdot\widetilde{b}$ and $1_{\widetilde{a+b}} = 1_{\widetilde{a}\cdot\overline{b}+\overline{a}\cdot\widetilde{b}}$.

$$
\begin{array}{llllll}
\widetilde{a+b} & \overset{e17}{=} & \widetilde{a}\cdot\widetilde{b} + \widetilde{a}\cdot\widehat{b} + \widehat{a}\cdot\widetilde{b} & = & \widetilde{a}\cdot\overline{b} + \overline{a}\cdot\widetilde{b} \\
1_{\widetilde{a+b}} & \overset{D2}{=} & 1_{a+b} & \overset{C2}{=} & 1_a + 1_b. \\
1_{\widetilde{a}\cdot\overline{b}+\overline{a}\cdot\widetilde{b}} & \overset{C1,C2}{=} & 1_{\widetilde{a}}\cdot 1_{\overline{b}} + 1_{\overline{a}}\cdot 1_{\widetilde{b}} & \overset{\text{def}}{=} & 1_{\widetilde{a}}\cdot 1_{\widehat{b}+\widetilde{b}} + 1_{\widehat{a}+\widetilde{b}}\cdot 1_{\widetilde{b}} \\
 & \overset{C2}{=} & 1_{\widetilde{a}}\cdot(1_{\widehat{b}} + 1_{\widetilde{b}}) + (1_{\widehat{a}} + 1_{\widetilde{a}})\cdot 1_{\widetilde{b}} & \overset{D3}{=} & 1_{\widetilde{a}}\cdot(1 + 1_{\widetilde{b}}) + (1 + 1_{\widetilde{a}})\cdot 1_{\widetilde{b}} \\
 & = & 1_{\widetilde{a}}\cdot 1 + 1\cdot 1_{\widetilde{b}} & = & 1_{\widetilde{a}} + 1_{\widetilde{b}} \\
 & \overset{D2}{=} & 1_a + 1_b & &
\end{array}
$$

Therefore $1_{\widetilde{a+b}} = 1_{\widetilde{a}\cdot\overline{b}+\overline{a}\cdot\widetilde{b}}$ and the second part of the lemma is proved.

3.7 Models of PPC_3^c

Definition 3.38 *A PPC_3^c-model is a pair $\langle \mathsf{B}, \delta\rangle$, where B is a Boolean algebra $\mathsf{B} = \langle B; \sqcap, \sqcup, \bot, \top, \overline{-}\rangle$ and δ is an assignment that maps every variable in the language to an element of the set*

$$M := \{\langle p, q\rangle \in B^2 \mid q \sqsubseteq p\}$$

where $\sqsubseteq$ indicates the order on B : $q \sqsubseteq p$ means $q \sqcap p = q$ or, equivalently, $q \sqcup p = p$.

Given a model we define the interpretation of every term of PPC_3^c by an element of M

$$[\![-]\!]_\delta : T \longrightarrow M$$

by induction on the structure of the term (the functions π_1 and π_2 are the first and second projection, respectively: $\pi_1\langle p, q\rangle = p$ and $\pi_2\langle p, q\rangle = q$):

$$
\begin{array}{lll}
[\![a]\!]_\delta & := & \delta(\alpha) \qquad \text{for every literal } \alpha \\
[\![\odot]\!]_\delta & := & \langle \bot, \bot\rangle \\
[\![1]\!]_\delta & := & \langle \top, \top\rangle \\
[\![\widetilde{a}]\!]_\delta & := & \langle \pi_1[\![a]\!]_\delta, \pi_1[\![a]\!]_\delta \sqcap \overline{\pi_2[\![a]\!]_\delta}\rangle \\
[\![1_a]\!]_\delta & := & \langle \top, \pi_1[\![a]\!]_\delta\rangle \\
[\![\blacktriangle_a]\!]_\delta & := & \langle \pi_2[\![a]\!]_\delta, \bot\rangle \\
[\![a+b]\!]_\delta & := & \langle \pi_1[\![a]\!]_\delta \sqcup \pi_1[\![b]\!]_\delta, \pi_2[\![a]\!]_\delta \sqcup \pi_2[\![b]\!]_\delta\rangle \\
[\![a\cdot b]\!]_\delta & := & \langle \pi_1[\![a]\!]_\delta \sqcap \pi_1[\![b]\!]_\delta, \pi_2[\![a]\!]_\delta \sqcap \pi_2[\![b]\!]_\delta\rangle
\end{array}
$$

The elementary relation $\equiv$ is interpreted as identity of the interpretations of the terms.

Definition 3.39 *$a \equiv b$ is* valid *in the PPC_3^c-model $\langle \mathsf{B}, \delta\rangle$ if $[\![a]\!]_\delta = [\![b]\!]_\delta$.*

Theorem 3.40 (Validity Theorem) *If $a \equiv b$ is provable in PPC_3^c then $[\![a]\!]_\delta = [\![b]\!]_\delta$ for every PPC_3-model $\langle \mathsf{B}, \delta\rangle$.*

Proof By induction on the length of the proof of $a \equiv b$. It is enough to prove the validity of all the axioms.

By the definition of the interpretation it follows that the defined symbols are interpreted in the following way:

$$
\begin{array}{lll}
[\![0]\!]_\delta & = & \langle \top, \bot\rangle \\
[\![\widehat{a}]\!]_\delta & = & \langle \top, \overline{\pi_1[\![a]\!]_\delta}\rangle \\
[\![\overline{a}]\!]_\delta & = & \langle \top, \overline{\pi_2[\![a]\!]_\delta}\rangle
\end{array}
$$

3.8 Equivalence with PPC_3-models

If we have a PPC_3^c-model (i.e. one of the models of Definition 3.38) we can obtain a PPC_3-model (i.e. a model in the sense of Definition 3.9) by taking the same Boolean algebra B and defining the maps ρ and ξ on the variables as

$$\begin{array}{rcl}\rho(\alpha) & := & \pi_2(\delta(\alpha)),\\ \xi(\alpha) & := & \pi_1(\delta(\alpha)).\end{array}$$

Vice versa given a PPC_3-model $\langle \mathsf{B}, \rho, \xi\rangle$ we obtain a PPC_3^c-model by taking the same Boolean algebra B and defining the map δ as

$$\delta(\alpha) := \langle \xi(\alpha), \rho(\alpha)\rangle.$$

3.9 Completeness of PPC_3^c

We prove now completeness of PPC_3^c with respect to the defined models, deriving it from the completeness of PPC_3 and the correspondence between the models of the two systems outlined in subsection 3.8.

Theorem 3.41 *Let a and b be two propositions. If for every PPC_3^c-model $\langle \mathsf{B}, \delta\rangle$, $[\![a]\!]_\delta = [\![b]\!]_\delta$, then $a \equiv b$ is derivable in PPC_3^c.*

Proof Suppose the interpretations of a and b coincide in every model. We construct a term model by taking the Boolean algebra $\mathcal{B} := (T/=)$ of terms of definition 3.14 and defining the assignment δ as $\delta(\alpha) := \langle \xi(\alpha), \rho(\alpha)\rangle = \langle [1_\alpha], [\alpha]\rangle$ for every atomic proposition α. By lemma 3.15 and subsection 3.8, $\langle \mathcal{B}, \delta\rangle$ is a model of PPC_3^c. Hence $[\![a]\!]_\delta = [\![b]\!]_\delta$ by hypothesis. We prove a preparatory lemma.

Lemma 3.42 *For every proposition a, $[\![a]\!]_\delta = \langle [\![1_a]\!]_{\rho\xi}, [\![a]\!]_{\rho\xi}\rangle$.*

Proof By induction on the structure of a.

Using the lemma we have that

$$\langle [\![1_a]\!]_{\rho\xi}, [\![a]\!]_{\rho\xi}\rangle = \langle [\![1_b]\!]_{\rho\xi}, [\![b]\!]_{\rho\xi}\rangle.$$

The two components must be equal, $[\![1_a]\!]_{\rho\xi} = [\![1_b]\!]_{\rho\xi}$ and $[\![a]\!]_{\rho\xi} = [\![b]\!]_{\rho\xi}$. By lemma 3.15 we have then that $1_a = 1_b$ and $a = b$, that is, $a \equiv b$ by theorem 3.36.

4 Further perspectives: modal logic

The concept of noncompositional operator can be put to further use, e.g. in the logic of the modalities POSSIBLE (Poss) and NECESSARY (Nec). Natural language modalities differ from metaphysical modalities in that they are valuated relative to a given context or knowledge state, representable as a given sentence A^G. $\mathsf{Poss}(B)$ means that B is consistent with A^G, and $\mathsf{Nec}(B)$ means that B is entailed by A^G.

More formally, for every given sentence A^G there is a set of new sentences $\mathbf{P}_{A^G}$, the sentences that are possible relative to A^G, defined as $\mathbf{P}_{A^G} := \{B\colon /A^G/ \cap /B/ \neq \emptyset\}$. If $B \in \mathbf{P}_{A^G}$, then $\mathsf{Poss}(B)$ is true relative to A^G.

For every given sentence A^G there is also a set of new sentences $\mathbf{N}_{A^G}$, the sentences that are necessary relative to A^G, defined as $\mathbf{N}_{A^G} := \{B\colon /A^G/ \subseteq /B/\}$. If $B \in \mathbf{N}_{A^G}$, then $\mathsf{Nec}(B)$ is true relative to A^G.

What are $/\mathsf{Poss}(B)/$ and $/\mathsf{Nec}(B)/$? Note that $\mathsf{Poss}(B)$ and $\mathsf{Nec}(B)$ are not sentences in the ordinary sense (where the interpretation of a sentence is the set of situations in which it is the case). The sentences $\mathsf{Poss}(B)$ and $\mathsf{Nec}(B)$ are just true or false and have no direct interpretation as a Σ-space. A key of propositions is required, i.e. a PARAKEY. (A METAKEY is a key of linguistic elements, not propositions.) The elements of the PARAKEY (**PK**) are discourse domains, i.e. propositions. The relation between modal propositions (e.g. $\mathsf{Poss}(B)$) and discourse domains (e.g. A^G in the previous case) parallels the one between ordinary propositions and states in the world. As we define the extension of an ordinary proposition A as the set of situations $\mathbf{s}$ that make A true, we can define the extension of a modal proposition as the set of discourse domains that make it true.

Hence the extension of $\mathsf{Poss}(B)$ is the set of all those discourse domains (propositions) A such that A makes $\mathsf{Poss}(B)$ true, that is, the set of those A such that B is possible relative to A:

$$\begin{aligned} /\mathsf{Poss}(B)/ &= \{A \mid A \text{ makes } \mathsf{Poss}(B) \text{ true}\} \\ &= \{A \mid B \text{ is consistent with } A\} \\ &= \{A \mid B \in \mathbf{P}_A\} \\ &= \{A \mid /A/ \cap /B/ \neq \emptyset\} \end{aligned}$$

as depicted in figure 16, where we call **PPK** the universe containing the second level (modal) propositions.

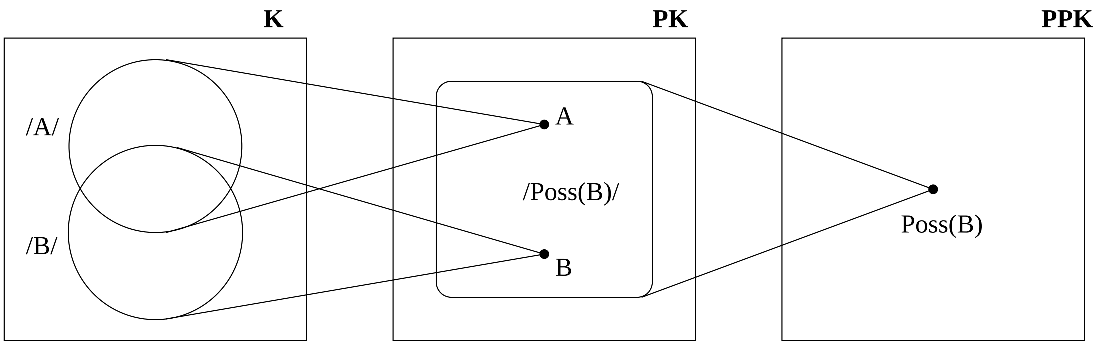


Figure 16: Set-theoretic interpretation of the modality of possibility.

Similarly the extension of $\mathsf{Nec}(B)$ is the set of all those discourse domains (propositions) A such that A makes $\mathsf{Nec}(B)$ true, that is, the set of those A such that B is necessary relative to A:

$$\begin{aligned} /\mathsf{Nec}(B)/ &= \{A \mid A \text{ makes } \mathsf{Nec}(B) \text{ true}\} \\ &= \{A \mid A \text{ entails } B\} \\ &= \{A \mid B \in \mathbf{N}_A\} \\ &= \{A \mid /A/ \subseteq /B/\} \end{aligned}$$

as depicted in figure 17.

Our arguments on presuppositions hold also at this second level, once we specify what the presuppositions of modal sentences are. Every proposition $\mathsf{Poss}(B)$ or $\mathsf{Nec}(B)$ presupposes that B is well-formed, well-anchored and well-keyed (i.e. has a TV). This happens when the presuppositions of B are fulfilled. Therefore the presupposition of $\mathsf{Poss}(B)$ (or of $\mathsf{Nec}(B)$) is satisfied whenever 1_B is true. However, we must be careful not to confuse the two levels: the extension of 1_B is a subset of **K**, whereas we expect the extension of $1_{\mathsf{Poss}(B)}$ to be a subset of **PK**. In other words the presupposition of $\mathsf{Poss}(B)$ cannot be 1_B, because the latter is an element of **PK**, whereas $1_{\mathsf{Poss}(B)}$ needs to be an element of **PPK**. $1_{\mathsf{Poss}(B)}$ should be a para-proposition whose extension consists of all the discourse domains in which B is well-keyed and well-anchored, i.e. all the discourse domains that entail 1_B. In conclusion we expect that

$$/1_{\mathsf{Poss}(B)}/ = \{A \mid A \text{ entails } 1_B\} = /\mathsf{Nec}(1_B)/$$

The natural definition is thus $1_{\mathsf{Poss}(B)} := \mathsf{Nec}(1_B)$. Similarly $1_{\mathsf{Nec}(B)} := \mathsf{Nec}(1_B)$.

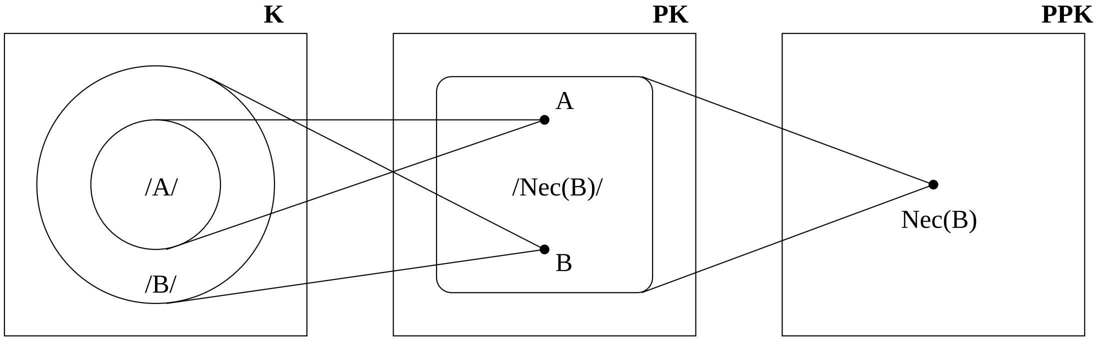
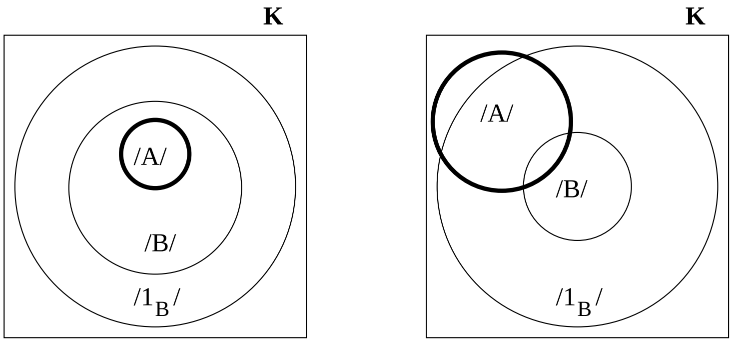


Figure 17: Set-theoretic interpretation of the modality of necessity.

This is not yet correct: the given definitions do not satisfy in general the property that for every proposition B, the extension of B is contained in the extension of its presupposition, $/B/ \subseteq /1_B/$ (fig. 18 left). The property holds for the necessity operator, $/\mathsf{Nec}(B)/ \subseteq /1_{\mathsf{Nec}(B)}/ = /\mathsf{Nec}(1_B)/$, for every proposition B; but it fails for the possibility operator, as it is not in general true that $/\mathsf{Poss}(B)/ = \{A \mid /A/ \cap /B/ \neq \emptyset\}$ is contained in $/1_{\mathsf{Poss}(B)}/ = /\mathsf{Nec}(1_B)/ = \{A \mid /A/ \subseteq /1_B/\}$ (fig. 18 right).



$A \in /\mathsf{Nec}(B)/ \Rightarrow A \in /\mathsf{Nec}(1_B)/$ $\quad$ $A \in /\mathsf{Poss}(B)/$ but $A \notin /\mathsf{Nec}(1_B)/$

Figure 18: Extension of the presuppositions of modal sentences.

We must therefore change the definition of $/\mathsf{Poss}(B)/$. The correct definition is

$$/\mathsf{Poss}(B)/ = \{A \mid /A/ \subseteq /1_B/ \text{ and } /A/ \cap /B/ \neq \emptyset\}.$$

Then it is the case that $/A/ \subseteq /1_A/$ holds in general. For example, for $A = \mathsf{Poss}(B)$ we find that $/1_{\mathsf{Poss}(B)}/ = /\mathsf{Nec}(1_B)/ = \{A \mid /A/ \subseteq /1_B/\}$, which is clearly a superset of $/\mathsf{Poss}(B)/$, according to the definition of $/\mathsf{Poss}(B)/$ that we have just given.

From these definitions the usual modal theorems $\mathsf{Nec}(\sim B) = \sim(\mathsf{Poss}(B))$ and $\mathsf{Poss}(\sim B) =$

$\sim$(Nec(B)) follow:

$$\begin{array}{rcl}
/\mathsf{Nec}(\sim B)/ & = & \{A \mid /A/ \subseteq /\sim B/\} \\
 & = & \{A \mid /A/ \subseteq /1_B/ - /B/\} \\
/\sim(\mathsf{Poss}(B))/ & = & /1_{\mathsf{Poss}(B)}/ - /\mathsf{Poss}(B)/ \\
 & = & /\mathsf{Nec}(1_B)/ - /\mathsf{Poss}(B)/ \\
 & = & \{A \mid /A/ \subseteq /1_B/\} - \{A \mid /A/ \subseteq /1_B/ \text{ and } /A/ \cap /B/ \neq \emptyset\} \\
 & = & \{A \mid /A/ \subseteq /1_B/ \text{ and } /A/ \cap /B/ = \emptyset\} \\
 & = & \{A \mid /A/ \subseteq /1_B/ - /B/\} \\
/\mathsf{Poss}(\sim B)/ & = & \{A \mid /A/ \subseteq /1_{\sim B}/ \text{ and } /A/ \cap /\sim B/ \neq \emptyset\} \\
 & = & \{A \mid /A/ \subseteq /1_B/ \text{ and } /A/ \cap (/1_B/ - /B/) \neq \emptyset\} \\
 & = & \{A \mid /A/ \subseteq /1_B/ \text{ and } /A/ - /B/ \neq \emptyset\} \\
/\sim(\mathsf{Nec}(B))/ & = & /1_{\mathsf{Nec}(B)}/ - /\mathsf{Nec}(B)/ \\
 & = & /\mathsf{Nec}(1_B)/ - /\mathsf{Nec}(B)/ \\
 & = & \{A \mid /A/ \subseteq /1_B/\} - \{A \mid /A/ \subseteq /B/\} \\
 & = & \{A \mid /A/ \subseteq /1_B/ \text{ and } /A/ \not\subseteq /B/\} \\
 & = & \{A \mid /A/ \subseteq /1_B/ \text{ and } /A/ - /B/ \neq \emptyset\}
\end{array}$$

Note that the modal theorems do *not* hold for the other negations:

$$\begin{array}{rcl}
/\mathsf{Nec}(\simeq B)/ & = & \{A \mid /A/ \subseteq \overline{/1_B/}\} \\
/\simeq \mathsf{Poss}(B)/ & = & \{A \mid /A/ \not\subseteq /1_B/\}
\end{array}$$

So, in general $/\mathsf{Nec}(\simeq B)/ \neq /\simeq \mathsf{Poss}(B)/$. Similarly for the Boolean negation.

References

Ballmer, T. T. (1979), Context change, truth and competence, *in* Bäuerle, Egli & von Stechow (1979), pp. 21–31.

Barwise, J. & Perry, J. (1983), *Situations and Attitudes*, MIT Press, Cambridge, MA.

Bäuerle, R., Egli, U. & von Stechow, A., eds (1979), *Semantics from Different Points of View*, Springer, Berlin–Heidelberg–New York.

Blau, U. (1978), *Die dreiwertige Logik der Sprache. Ihre Syntax, Semantik und Anwendung in der Sprachanalyse*, De Gruyter, Berlin–New York.

Boole, G. (1847), *The Mathematical Analysis of Logic*, Cambridge University Press, Cambridge.

Burley, W. (1988), *Von der Reinheit der Kunst der Logik. Erster Traktat. Von den Eigenschaften der Termini.* (= De puritate artis logicae. De proprietatibus terminorum), Felix Meiner Verlag, Hamburg. Translated and edited by Peter Kunze, with introduction and commentary.

Burton-Roberts, N. (1989), *The Limits to Debate. A Revised Theory of Semantic Presupposition*, Cambridge University Press, Cambridge.

Choon-Kyu, O. & Dinneen, D. A., eds (1979), *Presupposition*, Vol. 11 of *Syntax and Semantics*, Academic Press, New York–San Francisco–London.

Dummett, M. (1973), *Frege. Philosophy of language*, Duckworth, London.

Fauconnier, G. (1985), *Mental Spaces. Aspects of Meaning Construction in Natural Language*, MIT Press, Cambridge, MA.

Frege, G. (1892), 'Ueber sinn und bedeutung', *Zeitschrift für Philosophie und philosophische Kritik* **100**, 25–50.

Gazdar, G. (1979), *Pragmatics: Implicature, Presupposition, and Logical Form*, Academic Press, New York–San Francisco–London.

Geach, P. T. (1962), *Reference and Generality. An Examination of Some Medieval and Modern Theories*, Cornell University Press, Ithaca, New York.

Geach, P. T. (1969), Quine's syntactical insights, *in* D. Davidson & J. Hintikka, eds, 'Words and Objections. Essays on the work of W.V. Quine', Reidel, Dordrecht, pp. 146–157. (Also in Geach 1972, pp. 115–127).

Geach, P. T. (1972), *Logic Matters*, Blackwell, Oxford.

Geach, P. T. & Black, M. (1970), *Translations from the Philosophical Writings of Gottlob Frege*, Blackwell, Oxford.

Groenendijk, J. & Stokhof, M. (1991), 'Dynamic predicate logic', *Linguistics and Philosophy* **14**, 39–100.

Heim, I. R. (1982), The Semantics of Definite and Indefinite Noun Phrases, PhD thesis, University of Massachusetts, Amherst. Sonderforschungsbereich 99 Linguistik, Universität Konstanz.

Heim, I. R. (1983), On the projection problem for presuppositions, *in* 'Proceedings of the West Coast Conference on Formal Linguistics 2', pp. 114–126.

Horn, L. R. (1985), 'Metalinguistic negation and pragmatic ambiguity', *Language* **61**, 121–174.

Isard, S. (1975), Changing the context, *in* E. L. Keenan, ed., 'Formal Semantics and Natural Language', Cambridge University Press, Cambridge, pp. 287–296.

Kamp, H. (1981), A theory of truth and semantic interpretation, *in* J. A. G. Groenendijk, T. M. V. Janssen & M. B. J. Stokhof, eds, 'Formal Methods in the Study of Language', Vol. I, Mathematisch Centrum, Amsterdam, pp. 227–322.

Kamp, H. & Reyle, U. (1993), *From Discourse to Logic. Introduction to Model-Theoretic Semantics of Natural Language, Formal Logic and Discourse Representation Theory*, Kluwer, Dordrecht.

Karttunen, L. & Peters, S. (1979), Conventional implicature, *in* Choon-Kyu & Dinneen (1979), pp. 1–56.

Kleene, S. C. (1938), 'On notation for ordinal numbers', *Journal of Symbolic Logic* **3**, 150–155.

Kleene, S. C. (1952), *Introduction to Metamathematics*, North-Holland, Amsterdam.

Kneale, W. & Kneale, M. (1962), *The Development of Logic*, Clarendon Press, Oxford.

Landman, F. (1986), *Towards a Theory of Information. The Status of Partial Objects in Semantics*, Vol. 6 of *GRASS*, Foris, Dordrecht.

Lewis, D. (1979), Scorekeeping in a language game, *in* Bäuerle et al. (1979), pp. 172–187.

McCawley, J. D. (1979), Presupposition and discourse structure, *in* Choon-Kyu & Dinneen (1979), pp. 371–388.

Ogden, C. K. & Richards, I. A. (1923), *The Meaning of Meaning. A Study of the Influence of Language upon Thought and of the Science of Symbolism*, Routledge & Kegan Paul, London.

Quine, W. V. (1960), *Word and Object*, MIT Press, Cambridge, Mass.

Russell, B. (1905), 'On denoting', *Mind* **14**, 479–493.

Ryle, G. (1949), *The Concept of Mind*, Hutchinson, London.

Seuren, P. A. (1972), 'Taaluniversalia in de transformationele grammatica', *Leuvense Bijdragen* **61**(4), 311–370.

Seuren, P. A. (1975), *Tussen Taal en Denken. Een bijdrage tot de empirische funderingen van de semantiek*, Oosthoek, Scheltema & Holkema, Utrecht.

Seuren, P. A. (1977), 'Forme logique et forme sémantique: un argument contre M.Geach', *Logique et Analyse* **20**(79), 338–347.

Seuren, P. A. (1985), *Discourse Semantics*, Blackwell, Oxford.

Seuren, P. A. (1988), 'Presupposition and negation', *Journal of Semantics* **6**, 175–226.

Seuren, P. A. (1990), 'Burton-Roberts on presupposition and semantics', *Journal of Linguistics* **25**(2), 425–453.

Seuren, P. A. (1998*a*), Towards a discourse-semantic account of donkey anaphora, *in* S. Botley & A. McEnery, eds, 'New Approaches to Discourse Anaphora: Proceedings of the Second Colloquium on Discourse Anaphora and Anaphor Resolution (DAARC2)', University Centre for Computer Corpus Research on Language, pp. 212–220. Technical Papers, Vol. 11, Special Issue.

Seuren, P. A. (1998*b*), *Western Linguistics. An Historical Introduction*, Blackwell, Oxford.

Seuren, P. A. (2000), 'Presupposition, negation and trivalence', *Journal of Linguistics* **36**(2), 261–297.

Sidgwick, A. (1895), 'Context and meaning', *Mind* **4**, 281–306.

Stalnaker, R. (1973), 'Presuppositions', *Journal of Philosophical logic* **2**, 447–457.

Stegmüller, W. (1957), *Das Wahrheitsproblem und die Idee der Semantik. Eine Einleitung in die Theorien von A. Tarski und R. Carnap*, Springer, Vienna.

Strawson, P. F. (1950), 'On referring', *Mind* **59**, 320–344.

Strawson, P. F. (1952), *Introduction to logical theory*, Methuen, London.

Van den Auwera, J. (1979), Pragmatic presupposition: shared beliefs in a theory of irrefutable meaning, *in* Choon-Kyu & Dinneen (1979), pp. 249–264.

Van Fraassen, B. (1971), *Formal Semantics and Logic*, Macmillan, New York–London.

Wunderlich, D. (1979), Meaning and context-dependence, *in* Bäuerle et al. (1979), pp. 161–171.

Zadeh, L. A. (1975), 'Fuzzy logic and approximate reasoning', *Synthese* **30**, 407–428.