

1 Proof terms for generalized natural deduction

2 **Herman Geuvers**

3 Radboud University Nijmegen & Technical University Eindhoven, The Netherlands
4 herman@cs.ru.nl

5 **Tonny Hurkens**

6
7 hurkens@science.ru.nl

8 — Abstract —

9 In previous work it has been shown how to generate natural deduction rules for propositional
10 connectives from truth tables, both for classical and constructive logic. The present paper extends
11 this for the constructive case with proof-terms, thereby extending the Curry-Howard isomorphism
12 to these new connectives. A general notion of conversion of proofs is defined, both as a conversion
13 of derivations and as a reduction of proof-terms. It is shown how the well-known rules for natural
14 deduction (Gentzen, Prawitz) and general elimination rules (Schroeder-Heister, von Plato, and
15 others), and their proof conversions can be found as instances. As an illustration of the power of
16 the method, we give constructive rules for the nand logical operator (also called *Sheffer stroke*).

17 As usual, conversions come in two flavours: either a *detour conversion* arising from a *detour*
18 *convertibility*, where an introduction rule is immediately followed by an elimination rule, or a
19 *permutation conversion* arising from an *permutation convertibility*, an elimination rule nested
20 inside another elimination rule. In this paper, both are defined for the general setting, as con-
21 versions of derivations and as reductions of proof-terms. The properties of these are studied as
22 proof-term reductions. As one of the main contributions it is proved that detour conversion is
23 strongly normalizing and permutation conversion is strongly normalizing: no matter how one
24 reduces, the process eventually terminates. Furthermore, the combination of the two conversions
25 is shown to be weakly normalizing: one can always reduce away all convertibilities.

26 **2012 ACM Subject Classification** Theory of computation → Proof theory, Theory of compu-
27 tation → Type theory, Theory of computation → Constructive mathematics, Theory of compu-
28 tation → Functional constructs

29 **Keywords and phrases** constructive logic, natural deduction, detour conversion, permutation
30 conversion, normalization Curry-Howard isomorphism

31 **Digital Object Identifier** 10.4230/LIPIcs.TYPES.2017.3

32 **Acknowledgements** We thank Iris van der Giessen and the anonymous referees for their valuable
33 comments on the earlier version of this paper.

34 **1 Introduction**

35 Natural deduction rules come in various forms, where the tree format is the most well-known.
36 One either puts formulas A as the nodes and leaves of the tree, or sequents $\Gamma \vdash A$, where Γ
37 is a sequence or a finite set of formulas. Other formalisms use a linear format, using flags or
38 boxes to explicitly manage the open and discharged assumptions.

We [7] use a natural deduction in sequent calculus style, where in addition all rules have
a special form:

$$\frac{\dots \quad \Gamma \vdash A_i \quad \dots \quad \dots \quad \Gamma, A_j \vdash D \quad \dots}{\Gamma \vdash D}$$



© Herman Geuvers and Tonny Hurkens;

licensed under Creative Commons License CC-BY

23rd International Conference on Types for Proofs and Programs (TYPES 2017).

Editors: Andreas Abel, Fredrik Nordvall Forsberg, and Ambrus Kaposi; Article No. 3; pp. 3:1–3:40

Leibniz International Proceedings in Informatics

LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

3:2 Proof terms for generalized natural deduction

39 So if the conclusion of a rule is $\Gamma \vdash D$, then the hypotheses of the rule can be of one of two
40 forms:

- 41 1. $\Gamma, A_j \vdash D$: we still need to prove D from Γ , but we are now also allowed to use A_j as
42 additional assumption. We call A_j a case.
- 43 2. $\Gamma \vdash A_i$: in stead of proving D from Γ , we now need to prove A_i from Γ . We call A_i a
44 lemma.

Given the restricted format of the rules, we don't have to give Γ explicitly, as it can be retrieved from the other information in a deduction. So, the deduction rules are presented without Γ , in the following format

$$\frac{\dots \vdash A_i \quad \dots \quad \dots \quad A_j \vdash D \quad \dots}{\vdash D}$$

45 In [7] we have shown how to derive natural deduction rules for a connective from its
46 definition by a truth table, both for the classical and the intuitionistic case. In that paper,
47 we have shown that the intuitionistic rules are indeed constructive by providing a Kripke
48 semantics. In the present paper we provide a proof-theoretic study of the natural deduction
49 rules for the intuitionistic case. We define a notion of convertibility and conversion for the
50 general connectives, which we analyze by interpreting derivations as proof-terms. So we
51 extend the Curry-Howard isomorphism, that interprets formulas as types and derivations as
52 terms, to include all these new intuitionistic connectives.

53 It turns out that the standard format for the deduction rules we have chosen (as described
54 above) is very suitable for defining convertibilities and conversion in general, for giving a
55 term interpretation to derivations and for defining reductions on these proof-terms that
56 correspond with conversion (both detour conversion and permutation conversion). The format
57 of our rules also allows the transformation of other formalisms, like the very well-known
58 ones by Gentzen and Prawitz [6, 14] but also more recent ones by Von Plato [23], in terms
59 of ours. This transformation we will define on the proof-term level and we will show how
60 *detour conversion* (the elimination of a *direct convertibility*, an introduction rule immediately
61 followed by an elimination rule) is preserved by the translation.

62 Standard questions about logic are consistency and decidability. We prove that both
63 hold (in general for our connectives) by proving *weak normalization* for the combined
64 process of *detour conversion* and *permutation conversion*. A permutation conversion operates
65 on a *permutation convertibility*, which arises when an elimination rule blocks a detour
66 convertibility for another connective; in that case one has to permute one elimination
67 rule over another. Weak normalization states that for any derivation (proof-term) we can
68 eliminate convertibilities in such a way that eventually no convertibilities are left. Using this
69 one can prove the sub-formula property and consistency and decidability. We prove weak
70 normalization for the proof-terms by studying reduction of proof-terms.

71 The interest of our work lies in the fact that the natural deduction rules can be defined
72 and analyzed in such a generic way, capturing very many known instances of deduction
73 rules for intuitionistic logic, but also new deduction rules for new connectives. The key
74 concepts that make this work are our general rule format (described above) and the fact that
75 our system provides natural deduction rules for each connective *in isolation*: rules for one
76 connective do not use another connective. We will illustrate this by giving the **nand** operator
77 as an extended example. We describe its constructive derivation rules, as they arise from the
78 truth tables. These rules are self-contained, so they only refer to **nand** itself, and we show
79 how to interpret intuitionistic proposition logic in the logic with only **nand**. We also give the
80 proof-terms and reductions for **nand**.

81 1.1 Related work and contribution of the paper

82 Natural deduction has been studied extensively, since the original work by Gentzen [6], both
83 for classical and intuitionistic logic. Overviews can be found in [22] and [12]. Also the
84 generalization of natural deduction to include other connectives or allow different derivation
85 rules has been studied by various researchers. Notably, there is the work of Schroeder-Heister
86 [17], Read [16], Tennant [21], Von Plato [23, 12], Milne [11], Francez and Dyckhoff [4, 3] that
87 is related to ours. Schroeder-Heister studies general formats of natural deduction where also
88 rules may be discharged (as opposed to the normal situation where only formulas may be
89 discharged). He also studies a general rule format for intuitionistic logic and shows that
90 the connectives $\wedge, \vee, \rightarrow, \perp$ are complete for it. Von Plato, Milne, Francez and Dyckhoff,
91 Read and Tennant study “general elimination rules”, where the idea is that elimination
92 rules arise naturally from the introduction rules, following Prawitz’s [15] inversion principle:
93 “the conclusion obtained by an elimination does not state anything more than what must
94 have already been obtained if the major premise of the elimination was inferred by an
95 introduction”. The elimination rules obtained have the same flavor as the elimination rules
96 we derive from truth tables: the conclusion of elimination Φ is not a sub-formula of Φ , but a
97 general formula D , where there are additional hypothesis that connect Φ and D . For the
98 standard intuitionistic connectives the general elimination rules are quite close to ours, but
99 \wedge -elimination is slightly different. Von Plato [23], Lopez-Escobar [10] and Tennant [21] study
100 the standard intuitionistic connectives with general rules.

101 A difference is that we focus not so much on the rules but on the fact that we can define
102 different and new connectives constructively. In our work, we do not take the introduction
103 rules as primary, with the elimination rules defined from them, but we derive elimination
104 and introduction rules directly from the truth table. Then we optimize them, which can be
105 done in various ways, where we adhere to a fixed format for the rules. Many of the general
106 elimination rules, for example for \wedge , appear naturally as a consequence of our approach of
107 deriving the rules from the truth table.

108 The idea of deriving deduction rules from the truth table also occurs in the work of Milne
109 [11], but in a slightly different way: from the introduction rules, a truth table is derived
110 and then the classical elimination rules are derived from the truth table. For the if-then-else
111 connective, this amounts to classical rules equivalent to ours in [7], but not optimized. We
112 start from the truth table and derive rules for intuitionistic logic.

113 As remarked, the main contribution of this paper is a proof-theoretic analysis of our
114 system of generalized natural deduction via the Curry-Howard isomorphism that interprets
115 derivations as proof terms and conversions as reductions. We show that many known
116 conversions and reductions are captured by our approach and we prove general normalization
117 results. These is a lot of related work on the Curry-Howard isomorphism that our work rests
118 on, for which we refer to [18, 8].

119 The present paper builds on research reported in [7]. To make this paper self-contained,
120 we include the definitions and some basic results and examples from [7]: Section 2 repeats the
121 main definitions of [7] in slightly expanded form, where Section 2.1 adds the new example of
122 the `nand`-connective (Sheffer stroke), which is worked out in detail, especially the connection
123 between `nand`-logic and intuitionistic proposition logic. Section 3 defines detour conversion
124 and permutation conversion on derivations; the second is new. Section 4 defines the Curry-
125 Howard isomorphism for our general natural deduction format and gives (general) proof
126 terms for natural deductions and reduction rules on them. Section 5 shows how the general
127 rules relate to so called “optimized” rules, which are the ones that are known from the
128 literature for natural deduction and for proof-terms. Section 6 proves normalization results

129 for the calculi of proof-terms. Sections 4, 5, 6 are all new; Section 2.1 is largely new and
 130 Section 3 is partially new.

131 **2 Deriving constructive natural deduction rules from truth tables**

132 To make this paper self contained and to fix notions and notations, we recap the main
 133 definitions from [7] and explain in detail how the elimination and introduction rules for a
 134 connective are derived from its truth table. The elimination rules have the following form. Φ
 135 is the formula we eliminate. We have $\Phi = c(A_1, \dots, A_n)$ where c is a connective of arity n
 136 and $n = k + \ell$. The formula D is arbitrary.

$$\frac{\vdash \Phi \quad \vdash A_{i_1} \quad \dots \quad \vdash A_{i_k} \quad A_{j_1} \vdash D \quad \dots \quad A_{j_\ell} \vdash D}{\vdash D} \text{el}$$

So, $A_{i_1}, \dots, A_{i_k}, A_{j_1}, \dots, A_{j_\ell}$ are the direct subformulas of $\Phi = c(A_1, \dots, A_n)$, where some appear as “lemma” and others as “case” in the derivation rule. The (intuitionistic) introduction rules have the following form. Again, c is a connective of arity n , $\Phi = c(A_1, \dots, A_n)$ and $n = k + \ell$. (Of course, every rule has its own specific sequence $i_1, \dots, i_k, j_1, \dots, j_\ell$.)

$$\frac{\vdash A_{i_1} \quad \dots \quad \vdash A_{i_k} \quad A_{j_1} \vdash \Phi \quad \dots \quad A_{j_\ell} \vdash \Phi}{\vdash \Phi} \text{in}$$

137 For a concrete connective c , we derive the elimination and introduction rules from the
 138 truth table, as described in the following Definition, taken from [7].

139 **► Definition 1.** Given an n -ary connective c with a truth table t_c (with 2^n rows). We write
 140 $\varphi = c(p_1, \dots, p_n)$, where p_1, \dots, p_n are proposition letters and we write $\Phi = c(A_1, \dots, A_n)$,
 141 where A_1, \dots, A_n are arbitrary propositions. Each row of t_c gives rise to an elimination rule
 142 or an introduction rule for c in the following way.

$$\begin{array}{l} 143 \frac{p_1 \quad \dots \quad p_n \mid c(p_1, \dots, p_n)}{a_1 \quad \dots \quad a_n \mid 0} \mapsto \frac{\vdash \Phi \quad \dots \vdash A_j (\text{if } a_j = 1) \dots \quad \dots A_i \vdash D (\text{if } a_i = 0) \dots}{\vdash D} \text{el} \\ 144 \\ 145 \frac{p_1 \quad \dots \quad p_n \mid c(p_1, \dots, p_n)}{b_1 \quad \dots \quad b_n \mid 1} \mapsto \frac{\dots \vdash A_j (\text{if } b_j = 1) \dots \quad \dots A_i \vdash \Phi (\text{if } b_i = 0) \dots}{\vdash \Phi} \text{in} \end{array}$$

If $a_j = 1$ in t_c , then A_j occurs as a lemma in the rule; if $a_i = 0$ in t_c , then A_i occurs as a case. The rules are given in abbreviated form and it should be understood that all judgments can be used with an extended hypotheses set Γ . So the elimination rule in full reads as follows (where Γ is a set of propositions).

$$\frac{\Gamma \vdash \Phi \quad \dots \Gamma \vdash A_j (\text{if } a_j = 1) \dots \quad \dots \Gamma, A_i \vdash D (\text{if } a_i = 0) \dots}{\Gamma \vdash D} \text{el}$$

146 In an elimination rule, we call $\vdash \Phi$ the *major premise* and the other hypotheses of the rule
 147 we call the *minor premises*.

148 **► Definition 2.** Given a set of connectives $\mathcal{C} := \{c_1, \dots, c_n\}$, we define the *intuitionistic*
 149 *natural deduction system* for \mathcal{C} , $\text{IPC}_{\mathcal{C}}$, by the following derivation rules.

■ The *axiom rule*

$$\frac{}{\Gamma \vdash A} \text{axiom (if } A \in \Gamma)$$

150 ■ The elimination rules for the connectives in \mathcal{C} and the intuitionistic introduction rules for
 151 the connectives in \mathcal{C} , as given in Definition 1.

152 We write $\Gamma \vdash_{\mathcal{C}} A$ if $\Gamma \vdash A$ is derivable using the derivation rules of $\text{IPC}_{\mathcal{C}}$.

► **Example 3.**

A	B	$A \vee B$	$A \wedge B$	$A \rightarrow B$	$\neg A$
0	0	0	0	1	1
0	1	1	0	1	1
1	0	1	0	0	0
1	1	1	1	1	0

153 1. From the truth table for \vee we derive the following intuitionistic rules for \vee . We label the
154 rules by the relevant entries in the truth table.

$$\frac{\vdash A \vee B \quad A \vdash D \quad B \vdash D}{\vdash D} \vee\text{-el} \quad \frac{A \vdash A \vee B \quad \vdash B}{\vdash A \vee B} \vee\text{-in}_{01}$$

$$\frac{\vdash A \quad B \vdash A \vee B}{\vdash A \vee B} \vee\text{-in}_{10} \quad \frac{\vdash A \quad \vdash B}{\vdash A \vee B} \vee\text{-in}_{11}$$

155 These rules are all intuitionistically correct, as one can observe by inspection. We will
156 show that these are equivalent to the well-known intuitionistic rules. We will also show
157 how these rules can be optimized and be reduced to 1 elimination rule and 2 introduction
158 rules, which are the well-known ones.

2. From the truth table for \wedge we derive the following intuitionistic rules for \wedge , 3 elimination
rules and one introduction rule.

$$\frac{\vdash A \wedge B \quad A \vdash D \quad B \vdash D}{\vdash D} \wedge\text{-el}_{00} \quad \frac{\vdash A \wedge B \quad A \vdash D \quad \vdash B}{\vdash D} \wedge\text{-el}_{01}$$

$$\frac{\vdash A \wedge B \quad \vdash A \quad B \vdash D}{\vdash D} \wedge\text{-el}_{10} \quad \frac{\vdash A \quad \vdash B}{\vdash A \wedge B} \wedge\text{-in}$$

These rules are all intuitionistically correct, as one can observe by inspection. We will
show that these are equivalent to the well-known intuitionistic rules. We will also show
how these rules can be optimized and be reduced to 2 elimination rules and 1 introduction
rule, which are the well-known ones. The elimination rules for \wedge have a bit the flavor of
the so called “general elimination rules” of Schroeder-Heister [17] and Von Plato [23], in
the sense that we don’t derive A , respectively B , from $A \wedge B$, but an auxiliary conclusion
 D is derived. This rule, also called the *parallel elimination rule* by Tennant [21], is as
follows.

$$\frac{\vdash A \wedge B \quad A, B \vdash D}{\vdash D} \wedge\text{-el}^{\text{par}}$$

159 We will show that this rule can be derived from ours. See Definition 45 and Lemma 46,
160 where this is shown using proof-terms.

3. From the truth table for \neg we also derive the following rules for \neg , one elimination rule
and one introduction rule.

$$\frac{A \vdash \neg A}{\vdash \neg A} \neg\text{-in} \quad \frac{\vdash \neg A \quad \vdash A}{\vdash D} \neg\text{-el}$$

The elimination rule is familiar. For the introduction rule: to prove $\neg A$, one “only” has
to prove $\neg A$ from A , which may seem limited. The traditional \neg -in rule is the following.

$$\frac{A \vdash \neg B \quad A \vdash B}{\vdash \neg A} \neg\text{-in}^t$$

3:6 Proof terms for generalized natural deduction

161 The two \neg -introduction rules are equivalent, which we will show in detail (using proof
 162 terms) in Lemma 53. To derive \neg -in^t from \neg -in one also needs \neg -el, so we view \neg -in as
 163 more primitive than the traditional rule \neg -in^t.

As an example of the intuitionistic derivation rules for \neg we show that $A \vdash \neg\neg A$ is derivable:

$$\frac{\frac{A, \neg A \vdash \neg A \quad A, \neg A \vdash A}{A, \neg A \vdash \neg\neg A} \neg\text{-el}}{A \vdash \neg\neg A} \neg\text{-in}$$

164 4. From the truth table for \rightarrow we derive the following intuitionistic rules for \rightarrow .

$$\frac{A \vdash A \rightarrow B \quad B \vdash A \rightarrow B}{\vdash A \rightarrow B} \rightarrow\text{-in}_{00} \quad \frac{\vdash A \rightarrow B \quad \vdash A \quad B \vdash D}{\vdash D} \rightarrow\text{-el}$$

$$\frac{A \vdash A \rightarrow B \quad \vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}_{10} \quad \frac{\vdash A \quad \vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}_{11}$$

These rules are all intuitionistically correct, as one can verify by inspection. For example, for \rightarrow -in₀₁, observe that if $A \vdash A \rightarrow B$, then $\vdash A \rightarrow B$, so the second hypothesis is superfluous. Similarly for \rightarrow -in₁₁, the first hypothesis is superfluous. We will show that these rules are equivalent to the well-known intuitionistic rules. We will also show how these rules can be optimized and be reduced to 1 elimination rule and 2 introduction rules. These are not the well-known ones, because the well-known \rightarrow -in-rule does not fit into our scheme:

$$\frac{A \vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}$$

165 In this rule, both the conclusion is changed *and* an assumption (case) is added. In our
 166 system, each rule has the property that a hypothesis either adds an assumption or changes
 167 the conclusion (while retaining the same set of assumptions), and this “or” is exclusive.

168 We continue this section with some more basic properties and notions, most of which
 169 have been described briefly in [7]. We also introduce some further notation.

170 In the logic IPC_C (Definitions 1 and 2) we can freely reuse formulas and weaken the
 171 context, so the structural rules of contraction and weakening are wired into the system.
 172 Because weakening is used a lot, we formulate it as a Lemma. The proof is an immediate
 173 induction on the derivation.

174 ► **Lemma 4** (Weakening). *If $\Gamma \vdash A$ with derivation Π and $\Gamma \subseteq \Delta$, then $\Delta \vdash A$ with derivation*
 175 *Π .*

176 In natural deduction in tree format, the elimination of a detour convertibility involves
 177 *composition* of derivations: the placing of one derivation on top of another, replacing a
 178 discharged leaf A on top of a derivation tree (an assumption) by a derivation of A . In
 179 natural deduction in sequent calculus style, this amounts to replacing an axiom $\Gamma, A \vdash A$,
 180 that appears as the leaf of a derivation tree, by a derivation of $\Delta \vdash A$, where $\Delta \subset \Gamma$. We
 181 first define more precisely how the composition of derivation works in natural deduction in
 182 sequent calculus style.

183 ► **Lemma 5.** *If $\Delta, \varphi \vdash \psi$, and $\Gamma \vdash \varphi$, then $\Gamma, \Delta \vdash \psi$*

184 **Proof.** By induction on the derivation of $\Delta, \varphi \vdash \psi$, using weakening (Lemma 4). ◀

185 To be a bit more precise about what happens with the derivations in the proof of Lemma
 186 5, let Π be the derivation of $\Delta, \varphi \vdash \psi$. Then, due to the format of our rules:

187 ■ The only place in Π where the hypothesis φ is actually used is at a leaf of Π , in an
 188 instance of the (axiom) rule.

189 ■ Contexts can only grow when we walk upwards in a derivation, so these leaves are of the
 190 form $\Delta', \varphi \vdash \varphi$ for some $\Delta' \supseteq \Delta$.

191 We replace this leaf by Σ , the derivation of $\Gamma \vdash \varphi$. Due to weakening, this Σ is also a
 192 derivation of $\Gamma, \Delta' \vdash \varphi$, so Π with the leaves of the form $\Delta', \varphi \vdash \varphi$ replaced by Σ yields a
 193 correct derivation of $\Gamma, \Delta \vdash \psi$.

► **Notation 6.** If Π is a derivation of $\Delta, \varphi \vdash \psi$ and Σ is a derivation of $\Gamma \vdash \varphi$, then we have a derivation of $\Gamma, \Delta \vdash \psi$ that looks like this:

$$\begin{array}{c} \vdots \Sigma \quad \quad \quad \vdots \Sigma \\ \vdots \quad \quad \quad \vdots \\ \Gamma \vdash \varphi \quad \dots \quad \Gamma \vdash \varphi \\ \vdots \quad \quad \quad \vdots \\ \quad \quad \quad \Pi \\ \quad \quad \quad \vdots \\ \Gamma, \Delta \vdash \psi \end{array}$$

194 So in Π , every application of an (axiom) rule at a leaf, deriving $\Delta' \vdash \varphi$ for some $\Delta' \supseteq \Delta$ is
 195 replaced by a copy of a derivation Σ , which is also a derivation of $\Delta', \Gamma \vdash \varphi$.

196 The fact that we have weakening supports the following convention.

► **Convention 7.** In examples, to simplify derivations we will often use the following format for an elimination rule (and similarly for an introduction rule).

$$\frac{\Gamma_0 \vdash \Phi \quad \dots \Gamma_j \vdash A_j \text{ (if } a_j = 1) \dots \quad \dots \Gamma_i, A_i \vdash D \text{ (if } a_i = 0) \dots}{\cup_{k=0}^n \Gamma_k \vdash D} \text{el}$$

197 This prevents us from having to copy the full Γ from the conclusion to the hypotheses in a
 198 rule; we can limit ourselves to the parts of Γ that we need for that particular branch in the
 199 derivation.

200 We now recall from [7] two lemmas that allow to reduce the number of deduction rules:
 201 some rules can be taken together and one or more of the hypotheses can be dropped. For
 202 completeness, we give these lemmas again here (Lemma 9 and Lemma 12), with their proofs.
 203 First, we motivate Lemma 9 by looking at the example of the rules for \wedge (Example 3).

► **Example 8.** From the truth table we have derived the following 3 intuitionistic elimination rules for \wedge .

$$\begin{array}{c} \frac{\vdash A \wedge B \quad A \vdash D \quad B \vdash D}{\vdash D} \wedge\text{-el}_{00} \quad \frac{\vdash A \wedge B \quad A \vdash D \quad \vdash B}{\vdash D} \wedge\text{-el}_{01} \\ \\ \frac{\vdash A \wedge B \quad \vdash A \quad B \vdash D}{\vdash D} \wedge\text{-el}_{10} \end{array}$$

These rules can be reduced to the following 2 equivalent elimination rules. The index in the rule indicates where it originates from: $\wedge\text{-el}_{0_}$ is the combination of $\wedge\text{-el}_{00}$ and $\wedge\text{-el}_{01}$.

$$\frac{\vdash A \wedge B \quad A \vdash D}{\vdash D} \wedge\text{-el}_{0_} \quad \frac{\vdash A \wedge B \quad B \vdash D}{\vdash D} \wedge\text{-el}_{_0}$$

204 It can be shown that these sets of rules are equivalent. Here we only show the derivability
 205 of the $\wedge\text{-el}_{0_}$ rule from the rules $\wedge\text{-el}_{00}$ and $\wedge\text{-el}_{01}$. As usual, for notational simplicity we
 206 suppress the context Γ . Suppose we have derivations of $\vdash A \wedge B$ and of $A \vdash D$. Then we
 207 have the following derivation.

3:8 Proof terms for generalized natural deduction

$$\frac{\frac{\frac{\frac{B \vdash A \wedge B \quad B, A \vdash D \quad B \vdash B}{B \vdash D} \wedge\text{-el}_{01}}{\vdash A \wedge B \quad A \vdash D} \quad \vdash D}{\vdash D} \wedge\text{-el}_{00}}$$

208 Note that the third and fourth hypothesis come from the first and second through weakening,
209 and the fifth hypothesis is the axiom rule

210 The general method here is that we can replace two rules that only differ in one hypothesis,
211 which in one rule occurs as a lemma and in the other as a case, by one rule where the hypothesis
212 is removed. It will be clear that the Γ 's above are not relevant for the argument, so we will
213 not write these.

► **Lemma 9.** *A system with two derivation rules of the form*

$$\frac{\vdash A_1 \dots \vdash A_n \quad B_1 \vdash D \dots B_m \vdash D \quad A \vdash D}{\vdash D} \quad \frac{\vdash A_1 \dots \vdash A_n \quad \vdash A \quad B_1 \vdash D \dots B_m \vdash D}{\vdash D}$$

is equivalent to the system with these two rules replaced by

$$\frac{\vdash A_1 \dots \vdash A_n \quad B_1 \vdash D \dots B_m \vdash D}{\vdash D}$$

Proof. The implication from bottom to top is immediate. From top to bottom, suppose we have the two given rules. We now derive the bottom one. Assume we have derivations of $\vdash A_1, \dots, \vdash A_n, B_1 \vdash D, \dots, B_m \vdash D$. We now have the following derivation of $\vdash D$.

$$\frac{\frac{\frac{\frac{A \vdash A_1 \dots A \vdash A_n \quad A \vdash A \quad A, B_1 \vdash D \dots A, B_m \vdash D}{A \vdash D}}{\vdash A_1 \dots \vdash A_n \quad B_1 \vdash D \dots B_m \vdash D} \quad A \vdash D}{\vdash D}}$$

214

215 Lemma 9 can be applied to elimination and introduction rules. An application to
216 elimination rules is given in Example 8. We now give two applications to introduction rules.

217 ► **Example 10.** From the truth table we have derived the following 3 intuitionistic introduc-
218 tion rules for \vee .

$$\frac{A \vdash A \vee B \quad \vdash B}{\vdash A \vee B} \vee\text{-in}_{01} \quad \frac{\vdash A \quad B \vdash A \vee B}{\vdash A \vee B} \vee\text{-in}_{10} \quad \frac{\vdash A \quad \vdash B}{\vdash A \vee B} \vee\text{-in}_{11}$$

Using Lemma 9, these rules can be reduced to the following 2 equivalent introduction rules. (We could call $\vee\text{-inl}$ also $\vee\text{-in}_{-1}$, but we use a more informative and standard name: “in-left”.)

$$\frac{\vdash A}{\vdash A \vee B} \vee\text{-inl} \quad \frac{\vdash B}{\vdash A \vee B} \vee\text{-inr}$$

219 ► **Example 11.** Similar to \vee , we can optimize the introduction rules for \rightarrow . From the truth
220 table we have derived the following 3 intuitionistic introduction rules for \rightarrow .

$$\frac{A \vdash A \rightarrow B \quad B \vdash A \rightarrow B}{\vdash A \rightarrow B} \rightarrow\text{-in}_{00} \quad \frac{A \vdash A \rightarrow B \quad \vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}_{01} \quad \frac{\vdash A \quad \vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}_{11}$$

Using Lemma 9, these rules can be reduced to the following 2 equivalent introduction rules.

$$\frac{A \vdash A \rightarrow B}{\vdash A \rightarrow B} \rightarrow\text{-in}_a \quad \frac{\vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}_b$$

It can easily be shown that the rules $\rightarrow\text{-in}_a$ and $\rightarrow\text{-in}_b$ together are equivalent with the well-known $\rightarrow\text{-in}$:

$$\frac{A \vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}$$

221 NB. To derive $\rightarrow\text{-in}_a$ from $\rightarrow\text{-in}$, one also needs $\rightarrow\text{-el}$.

222 As $\rightarrow\text{-in}$ does not conform with our format for rules, we will be using $\rightarrow\text{-in}_a$ and $\rightarrow\text{-in}_b$
 223 as our basic rules and treat $\rightarrow\text{-in}$ as a defined rule, the composition of first $\rightarrow\text{-in}_b$ and then
 224 $\rightarrow\text{-in}_a$.

Another optimization we can perform is to replace a rule which has only one case by a rule where the case is the conclusion. To illustrate this: the simplified elimination rules for \wedge , $\wedge\text{-el}_0$ and $\wedge\text{-el}_1$ have only one case. The rule $\wedge\text{-el}_0$ can thus be replaced by the rule $\wedge\text{-ell}$, which is the usual left projection rule, \wedge -elimination-left.

$$\frac{\vdash A \wedge B \quad A \vdash D}{\vdash D} \wedge\text{-el}_0 \quad \frac{\vdash A \wedge B}{\vdash A} \wedge\text{-ell}$$

225 There is a general Lemma stating this simplification is correct.

► **Lemma 12.** *A system with a derivation rule of the form to the left is equivalent to the system with this rule replaced by the rule on the right.*

$$\frac{\vdash A_1 \dots \vdash A_n \quad B \vdash D}{\vdash D} \quad \frac{\vdash A_1 \dots \vdash A_n}{\vdash B}$$

226 **Proof.** The implication from left to right is immediate. From right to left, assume we have
 227 derivations of $\vdash A_1, \dots, \vdash A_n$. Then, by the rule to the right, we have $\vdash B$. Now assume
 228 we also have a derivation of $B \vdash D$. By Lemma 5, we also have a derivation of $\vdash D$.

229 ◀

230 ► **Definition 13.** The *standard derivation rules* for the intuitionistic propositional connectives
 231 $\wedge, \vee, \rightarrow, \neg, \perp$ and \top are given below. These rules are derived from the truth tables and
 232 optimized following Lemmas 9 and 12. We have seen most of the rules in previous Examples,
 233 except for the rules for \top and \perp , which are derived immediately from Definition 1. The
 234 system with these connectives and rules we will call *intuitionistic proposition logic* and if we
 235 want to explicit we write $\Gamma \vdash_i A$ for derivability in this system.

$\frac{\vdash A \quad \vdash B}{\vdash A \wedge B} \wedge\text{-in}$	$\frac{\vdash A \wedge B}{\vdash A} \wedge\text{-ell}$	$\frac{\vdash A \wedge B}{\vdash B} \wedge\text{-elr}$
$\frac{\vdash A}{\vdash A \vee B} \vee\text{-inl}$	$\frac{\vdash B}{\vdash A \vee B} \vee\text{-inr}$	$\frac{\vdash A \vee B \quad A \vdash D \quad B \vdash D}{\vdash D} \vee\text{-el}$
$\frac{A \vdash A \rightarrow B}{\vdash A \rightarrow B} \rightarrow\text{-in}_a$	$\frac{\vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}_b$	$\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B} \rightarrow\text{-el}$
$\frac{A \vdash \neg A}{\vdash \neg A} \neg\text{-in}$	$\frac{\vdash \neg A \quad \vdash A}{\vdash D} \neg\text{-el}$	$\frac{}{\vdash \top} \top\text{-in} \quad \frac{\vdash \perp}{\vdash D} \perp\text{-el}$

236 **2.1 Three larger examples**

237 As examples we look in more detail at two ternary connectives and one binary connective.
 238 The ternary connectives we treat are *if-then-else*, the “if-then-else” connective, and *most*,
 239 the ternary connective that is true if at least 2 of the arguments are true. These have been
 240 discussed in finer detail in [7], notably the connective *if-then-else*. The binary connective
 241 that we study at the end of this section is the *nand*, written $A \uparrow B$ for $\text{nand}(A, B)$. It is
 242 also known as the *Sheffer stroke*, the well-known connective that is functionally complete
 243 classically, where $A \uparrow B$ expresses $\neg(A \wedge B)$.

244 The truth tables of *most* and *if-then-else* are as follows, where we denote *if A then B else C*
 245 by $A \rightarrow B/C$.

A	B	C	$\text{most}(A, B, C)$	$A \rightarrow B/C$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	0
1	1	0	1	1
1	1	1	1	1

246
 247 From the lines in the truth table of $A \rightarrow B/C$ with a 0 we get the following four elimination
 248 rules.

$\frac{\vdash A \rightarrow B/C \quad A \vdash D \quad B \vdash D \quad C \vdash D}{\vdash D}$	$\frac{\vdash A \rightarrow B/C \quad A \vdash D \quad \vdash B \quad C \vdash D}{\vdash D}$
$\frac{\vdash A \rightarrow B/C \quad \vdash A \quad B \vdash D \quad C \vdash D}{\vdash D}$	$\frac{\vdash A \rightarrow B/C \quad \vdash A \quad B \vdash D \quad \vdash C}{\vdash D}$

252 Using Lemmas 9 and 12, these can be reduced to the following two. (The two rules on
 253 the first line reduce to *else-el*, the two rules on the second line reduce to *then-el*.)

$\frac{\vdash A \rightarrow B/C \quad A \vdash D \quad C \vdash D}{\vdash D} \quad \text{else-el}$	$\frac{\vdash A \rightarrow B/C \quad \vdash A}{\vdash B} \quad \text{then-el}$
----------------------------------------------------------------------------------------------------	---------------------------------------------------------------------------------

These are not the only possible optimizations: the two rules on the left can also be combined into an “if-el” rule:

$$\frac{\vdash A \rightarrow B/C \quad B \vdash D \quad C \vdash D}{\vdash D} \quad \text{if-el}$$

254 From the lines in the truth table of $A \rightarrow B/C$ with a 1 we get the following four introduction
 255 rules:

$\frac{A \vdash A \rightarrow B/C \quad B \vdash A \rightarrow B/C \quad \vdash C}{\vdash A \rightarrow B/C}$	$\frac{A \vdash A \rightarrow B/C \quad \vdash B \quad \vdash C}{\vdash A \rightarrow B/C}$
$\frac{\vdash A \quad \vdash B \quad C \vdash A \rightarrow B/C}{\vdash A \rightarrow B/C}$	$\frac{\vdash A \quad \vdash B \quad \vdash C}{\vdash A \rightarrow B/C}$

259 Using Lemmas 9 and 12 can be reduced to the following two. (The two rules on the first
 260 line reduce to *else-in*, the two rules on the second line reduce to *then-in*.)

$$\boxed{\frac{A \vdash A \rightarrow B/C \quad \vdash C}{\vdash A \rightarrow B/C} \text{ else-in} \quad \frac{\vdash A \quad \vdash B}{\vdash A \rightarrow B/C} \text{ then-in}}$$

261 Again, these are not the only possible optimizations: the two rules on the right can also
262 be combined into an “if-in” rule:

$$\frac{\vdash B \quad \vdash C}{\vdash A \rightarrow B/C} \text{ if-in}$$

263 In [7], we have studied the if-then-else connective in more detail, and we have shown
264 that if-then-else, together with \top and \perp is *functionally complete*: all other constructive
265 connectives can be defined in terms of it.

266 From the lines in the truth table of $\text{most}(A, B, C)$ with a 0 we get the following four
267 elimination rules.

$$\begin{array}{c} \frac{\vdash \text{most}(A, B, C) \quad A \vdash D \quad B \vdash D \quad C \vdash D}{\vdash D} \\ \frac{\vdash \text{most}(A, B, C) \quad A \vdash D \quad \vdash B \quad C \vdash D}{\vdash D} \\ \frac{\vdash \text{most}(A, B, C) \quad A \vdash D \quad B \vdash D \quad \vdash C}{\vdash D} \\ \frac{\vdash \text{most}(A, B, C) \quad \vdash A \quad B \vdash D \quad C \vdash D}{\vdash D} \end{array}$$

271 Using Lemmas 9 and 12, these can be reduced to the following three. If we would follow
272 the naming conventions that we introduced earlier, we would have $\text{most-el}_1 = \text{most-el}_{00}$,
273 $\text{most-el}_2 = \text{most-el}_{0_0}$ and $\text{most-el}_3 = \text{most-el}_{_00}$, but we will not pursue that naming here.

$$\boxed{\begin{array}{c} \frac{\vdash \text{most}(A, B, C) \quad A \vdash D \quad B \vdash D}{\vdash D} \text{ most-el}_1 \quad \frac{\vdash \text{most}(A, B, C) \quad A \vdash D \quad C \vdash D}{\vdash B} \text{ most-el}_2 \\ \frac{\vdash \text{most}(A, B, C) \quad B \vdash D \quad C \vdash D}{\vdash B} \text{ most-el}_3 \end{array}}$$

274 From the lines in the truth table of $\text{most}(A, B, C)$ with a 1 we get the following four
275 introduction rules:

$$\begin{array}{c} \frac{A \vdash \text{most}(A, B, C) \quad \vdash B \quad \vdash C}{\vdash \text{most}(A, B, C)} \quad \frac{\vdash A \quad B \vdash \text{most}(A, B, C) \quad \vdash C}{\vdash \text{most}(A, B, C)} \\ \frac{\vdash A \quad \vdash B \quad C \vdash \text{most}(A, B, C)}{\vdash \text{most}(A, B, C)} \quad \frac{\vdash A \quad \vdash B \quad \vdash C}{\vdash \text{most}(A, B, C)} \end{array}$$

279 Using Lemmas 9 and 12 can be reduced to the following three.

$$\boxed{\frac{\vdash A \quad \vdash B}{\vdash \text{most}(A, B, C)} \text{ most-in}_1 \quad \frac{\vdash A \quad \vdash C}{\vdash \text{most}(A, B, C)} \text{ most-in}_2 \quad \frac{\vdash B \quad \vdash C}{\vdash \text{most}(A, B, C)} \text{ most-in}_3}$$

The truth table for $\text{nand}(A, B)$, which we write as $A \uparrow B$ is as follows.

A	B	$A \uparrow B$
0	0	1
0	1	1
1	0	1
1	1	0

3:12 Proof terms for generalized natural deduction

280 From this we derive the following 3 introduction and 1 elimination rule

$$\frac{A \vdash A \uparrow B \quad B \vdash A \uparrow B}{\vdash A \uparrow B} \uparrow\text{-in}_{00} \quad \frac{A \vdash A \uparrow B \quad \vdash B}{\vdash A \uparrow B} \uparrow\text{-in}_{01}$$

$$\frac{\vdash A \quad B \vdash A \uparrow B}{\vdash A \uparrow B} \uparrow\text{-in}_{10} \quad \frac{\vdash A \uparrow B \quad \vdash A \quad \vdash B}{\vdash D} \uparrow\text{-el}$$

281 The three introduction rules can be combined to two rules, so our optimized set of
282 deduction rules for **nand** consists of three rules. We call this **nand-logic**.

283 ► **Definition 14.** The logic with just the connective **nand** and the three derivation rules
284 below we define as **nand-logic**. We denote derivability in this logic by $\Gamma \vdash_{\uparrow} A$.

$\frac{A \vdash A \uparrow B}{\vdash A \uparrow B} \uparrow\text{-inl} \quad \frac{B \vdash A \uparrow B}{\vdash A \uparrow B} \uparrow\text{-inr} \quad \frac{\vdash A \uparrow B \quad \vdash A \quad \vdash B}{\vdash D} \uparrow\text{-el}$

285 We can define the usual connectives of intuitionistic proposition logic (Definition 13) in
286 terms of **nand** in the usual way. This gives rise to an embedding of intuitionistic proposition
287 logic into the **nand-logic**.

► **Definition 15.**

288 $\neg A := A \uparrow A$
289 $A \vee B := (A \uparrow A) \uparrow (B \uparrow B)$
290 $A \wedge B := (A \uparrow B) \uparrow (A \uparrow B)$
291 $A \rightarrow B := A \uparrow (B \uparrow B)$

292 This gives rise to the following interpretation of intuitionistic proposition logic into **nand-logic**.

293 $p^{\uparrow} := \neg\neg p$ for p proposition letter
294 $(\neg A)^{\uparrow} := \neg A^{\uparrow}$
295 $(A \vee B)^{\uparrow} := A^{\uparrow} \vee B^{\uparrow}$
296 $(A \wedge B)^{\uparrow} := A^{\uparrow} \wedge B^{\uparrow}$
297 $(A \rightarrow B)^{\uparrow} := A^{\uparrow} \rightarrow B^{\uparrow}$

298 This interpretation extends straightforwardly to sets of propositions.

299 As a side remark, the translation of a proposition letter p could also be chosen to be p in
300 stead of $\neg\neg p$. Then the soundness statement below (Proposition 17) requires an additional
301 double negation: If $\Gamma \vdash_i A$, then $\Gamma^{\uparrow} \vdash_{\uparrow} \neg\neg A^{\uparrow}$. The connective \uparrow is very much a “negative
302 connective” and the choice of $\neg\neg p$ as translation of p renders all formulas A^{\uparrow} negative, so
303 the double negation can be avoided.

304 Before proving the soundness of the interpretation we give some auxiliary lemmas.

305 ► **Lemma 16.** In **nand-logic**, we have the following.

1. For arbitrary propositions A and B ,

$$\neg\neg(A \uparrow B) \vdash A \uparrow B,$$

2. For every A ,

$$\neg\neg\neg A \vdash \neg A.$$

3. For every proposition P from intuitionistic proposition logic,

$$\dot{\neg}\dot{\neg}P^\uparrow \vdash P^\uparrow.$$

4. For arbitrary propositions A and B ,

$$\text{If } \Gamma, A \vdash B \text{ then } \Gamma, \dot{\neg}B \vdash \dot{\neg}A.$$

Proof. The following proves $\dot{\neg}\dot{\neg}(A \uparrow B) \vdash A \uparrow B$. Here $\Gamma = \dot{\neg}\dot{\neg}(A \uparrow B), A, B, A \uparrow B$ and the last \uparrow -in rule denotes a successive application of \uparrow -inl followed by \uparrow -inr. Finally, the lowest \uparrow -el has one premise more, which is an exact copy of the derivation of $\dot{\neg}\dot{\neg}(A \uparrow B), A, B \vdash \dot{\neg}(A \uparrow B)$ that is given.

$$\frac{\frac{\frac{\Gamma \vdash A \uparrow B \quad \Gamma \vdash A \quad \Gamma \vdash B}{\dot{\neg}\dot{\neg}(A \uparrow B), A, B, A \uparrow B \vdash \dot{\neg}(A \uparrow B)} \uparrow\text{-el}}{\dot{\neg}\dot{\neg}(A \uparrow B), A, B \vdash \dot{\neg}\dot{\neg}(A \uparrow B)} \uparrow\text{-in}}{\dot{\neg}\dot{\neg}(A \uparrow B), A, B \vdash A \uparrow B} \uparrow\text{-el}}{\dot{\neg}\dot{\neg}(A \uparrow B) \vdash A \uparrow B} \uparrow\text{-in}$$

306 So, $\dot{\neg}\dot{\neg}\dot{\neg}A \vdash \dot{\neg}A$ follows immediately, and similarly $\dot{\neg}\dot{\neg}P^\uparrow \vdash P^\uparrow$ for every proposition P from
307 intuitionistic proposition logic.

Now, assuming that $\Gamma, A \vdash B$, we can make the following derivation of $\Gamma, \dot{\neg}B \vdash \dot{\neg}A$, using the fact that $\Gamma, B \uparrow B, A \vdash B$ by weakening.

$$\frac{\frac{\Gamma, B \uparrow B, A \vdash B \uparrow B \quad \Gamma, B \uparrow B, A \vdash B \quad \Gamma, B \uparrow B, A \vdash B}{\Gamma, B \uparrow B, A \vdash A \uparrow A} \uparrow\text{-el}}{\Gamma, B \uparrow B \vdash A \uparrow A} \uparrow\text{-in}$$

308

◀

309 We can now prove the soundness of the interpretation of intuitionistic proposition logic
310 into **nand**-logic.

311 ▶ **Proposition 17.** If $\Gamma \vdash_i A$, then $\Gamma^\uparrow \vdash_\uparrow A^\uparrow$.

312 **Proof.** The proof is by induction on the derivation of $\Gamma \vdash_i A$, so we have to show that the
313 rules of intuitionistic proposition logic are sound inside **nand**-logic (after interpretation). We
314 use Lemma 16, notably case (4), which we indicate explicitly in the derivations.

■ \neg -in: we show that \neg -in of Definition 13 is derivable.

$$\frac{A \vdash A \uparrow A}{A \vdash A \uparrow A} \uparrow\text{-in}$$

■ \neg -el: we show that \neg -el of Definition 13 is derivable.

$$\frac{\vdash A \uparrow A \quad \vdash A \quad \vdash A}{\vdash D} \uparrow\text{-el}$$

■ \vee -in: we show that $A \vdash_\uparrow A \dot{\vee} B$ is derivable.

$$\frac{\frac{A, A \uparrow A \vdash A \uparrow A \quad A, A \uparrow A \vdash A \quad A, A \uparrow A \vdash A}{A, A \uparrow A \vdash (A \uparrow A) \uparrow (B \uparrow B)} \uparrow\text{-el}}{A \vdash (A \uparrow A) \uparrow (B \uparrow B)} \uparrow\text{-inl}$$

3:14 Proof terms for generalized natural deduction

- \vee -el: we show that the following rule is derivable (which suffices).

$$\frac{\frac{\frac{\frac{\frac{\frac{\vdash A \dot{\vee} B \quad A \vdash D \quad B \vdash D}{\vdash \dot{\vee} D}}{\vdash (A \uparrow A) \uparrow (B \uparrow B)} \quad \frac{\frac{A \vdash D}{D \uparrow D \vdash A \uparrow A} \text{16(4)} \quad \frac{B \vdash D}{D \uparrow D \vdash B \uparrow B} \text{16(4)}}{D \uparrow D \vdash (D \uparrow D) \uparrow (D \uparrow D)} \uparrow\text{-el}}{\vdash (D \uparrow D) \uparrow (D \uparrow D)} \uparrow\text{-inl}}{\vdash (D \uparrow D) \uparrow (D \uparrow D)} \uparrow\text{-inl}}$$

- 315 ■ \wedge -el: we show that $A \wedge B \vdash_{\uparrow} \dot{\vee} A$ is derivable.

$$\frac{\frac{\frac{\frac{\frac{\frac{A \uparrow A \vdash A \uparrow A \quad A \vdash A}{A \uparrow A, A \vdash A \uparrow B} \uparrow\text{-el}}{A \wedge B \vdash (A \uparrow B) \uparrow (A \uparrow B)} \uparrow\text{-inl}}{A \wedge B, A \uparrow A \vdash A} \uparrow\text{-el}}{\frac{A \wedge B, A \uparrow A \vdash (A \uparrow A) \uparrow (A \uparrow A)}{\vdash (A \uparrow A) \uparrow (A \uparrow A)} \text{16(4)}}{\vdash (A \uparrow A) \uparrow (A \uparrow A)} \uparrow\text{-inl}}$$

- \wedge -in: we show that the following rule is derivable (which suffices).

$$\frac{\frac{\frac{\frac{\frac{\frac{\vdash A \quad \vdash B}{\vdash A \wedge B}}{A \uparrow B \vdash A \uparrow B} \quad \vdash A \quad \vdash B}{A \uparrow B \vdash (A \uparrow B) \uparrow (A \uparrow B)} \uparrow\text{-el}}{\vdash (A \uparrow B) \uparrow (A \uparrow B)} \uparrow\text{-inl}}{\vdash (A \uparrow B) \uparrow (A \uparrow B)} \uparrow\text{-inl}}$$

- \rightarrow -in: we show that the following rule is derivable (which suffices).

$$\frac{\frac{\frac{\frac{\frac{\frac{A \vdash B}{\vdash A \rightarrow B}}{B \uparrow B \vdash B \uparrow B} \quad A \vdash B \quad A \vdash B}{A, B \uparrow B \vdash A \uparrow (B \uparrow B)} \uparrow\text{-el}}{A \vdash A \uparrow (B \uparrow B)} \uparrow\text{-inr}}{\vdash A \uparrow (B \uparrow B)} \uparrow\text{-inl}}$$

- \rightarrow -el: we show that the following rule is derivable (which suffices).

$$\frac{\frac{\frac{\frac{\frac{\frac{\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash \dot{\vee} B}}{\vdash A \uparrow (B \uparrow B)} \quad \vdash A \quad B \uparrow B \vdash B \uparrow B}{B \uparrow B \vdash B} \uparrow\text{-el}}{B \uparrow B \vdash (B \uparrow B) \uparrow (B \uparrow B)} \text{16(4)}}{\vdash (B \uparrow B) \uparrow (B \uparrow B)} \uparrow\text{-inl}}{\vdash (B \uparrow B) \uparrow (B \uparrow B)} \uparrow\text{-inl}}$$

316

317 The reverse of Proposition 17 does not hold. For example, $\not\vdash p \vee \neg p$, for p a proposition
 318 letter, while $(p \vee \neg p)^\uparrow = (\dot{p} \uparrow \dot{p}) \uparrow (\dot{\neg} \dot{p} \uparrow \dot{\neg} \dot{p})$, where $\dot{p} := \dot{\neg} \dot{\neg} p$. The proposition $(A \uparrow A) \uparrow$
 319 $(\dot{\neg} A \uparrow \dot{\neg} A)$ is derivable in **nand**-logic for any A (note that $\dot{\neg} A = A \uparrow A$):

$$\frac{\frac{\dot{\neg} A \uparrow \dot{\neg} A \vdash \dot{\neg} A \uparrow \dot{\neg} A \quad A \uparrow A \vdash \dot{\neg} A \quad A \uparrow A \vdash \dot{\neg} A}{A \uparrow A, \dot{\neg} A \uparrow \dot{\neg} A \vdash (A \uparrow A) \uparrow (\dot{\neg} A \uparrow \dot{\neg} A)} \uparrow\text{-el}}{\vdash (A \uparrow A) \uparrow (\dot{\neg} A \uparrow \dot{\neg} A)} \uparrow\text{-in}$$

320 There is also an obvious mapping from **nand**-logic to intuitionistic proposition logic, by
 321 interpreting $A \uparrow B$ as $\neg(A \wedge B)$. As a matter of fact, it can also be shown in the joint system
 322 (i.e. where we add **nand** to intuitionistic proposition logic) that $A \uparrow B$ and $\neg(A \wedge B)$ are
 323 equivalent: $A \uparrow B \vdash \neg(A \wedge B)$ and $\neg(A \wedge B) \vdash A \uparrow B$. In presence of the implication and
 324 conjunction connective, the latter can be reformulated as $\vdash A \uparrow B \longleftrightarrow \neg(A \wedge B)$ (where, as
 325 usual, we let $C \longleftrightarrow D$ abbreviate $(C \rightarrow D) \wedge (D \rightarrow C)$).

► **Definition 18.** We define the mapping $(-)^{\downarrow}$ from **nand**-logic to intuitionistic proposition logic by defining

$$(A \uparrow B)^{\downarrow} := \neg(A^{\downarrow} \wedge B^{\downarrow})$$

326 and further by induction on propositions. This mapping extends to sets of hypotheses Γ in
 327 the obvious way.

328 ► **Proposition 19.** If $\Gamma \vdash_{\uparrow} A$, then $\Gamma^{\downarrow} \vdash_i A^{\downarrow}$.

Proof. By induction on the derivation. The only thing to show is that the rules $\uparrow\text{-el}$, $\uparrow\text{-inl}$
 and $\uparrow\text{-inr}$ are sound in intuitionistic proposition logic if we interpret $A \uparrow B$ as $\neg(A \wedge B)$. So
 we have to verify the soundness of the following rules.

$$\frac{A \vdash \neg(A \wedge B)}{\vdash \neg(A \wedge B)} \quad \frac{B \vdash \neg(A \wedge B)}{\vdash \neg(A \wedge B)} \quad \frac{\vdash \neg(A \wedge B) \quad \vdash A \quad \vdash B}{\vdash D}$$

329 A simple inspection shows that these rules are sound in intuitionistic proposition logic. ◀

330 We can now formulate a Glivenko-like theorem that relates **nand**-logic and intuitionistic
 331 proposition logic. (Glivenko's theorem, e.g. see [22], relates intuitionistic and classical
 332 proposition logic via the double negation.)

► **Proposition 20.** For A a proposition of intuitionistic proposition logic,

$$\vdash_i A^{\uparrow\downarrow} \longleftrightarrow \neg\neg A$$

333 .

334 **Proof.** By induction on the structure of A .

- 335 ■ $A = p$, a proposition letter. Then $p^{\uparrow\downarrow} = (\dot{\neg} \dot{\neg} p)^{\downarrow} = \neg(\neg(p \wedge p) \wedge \neg(p \wedge p)) \longleftrightarrow \neg\neg p$.
- 336 ■ $A = \neg B$. Then $(\neg B)^{\uparrow\downarrow} = (B \uparrow B)^{\downarrow} = \neg(B \wedge B) \longleftrightarrow \neg\neg\neg B$.
- 337 ■ $A = B \vee C$. Then $(B \vee C)^{\uparrow\downarrow} = ((B \uparrow B) \uparrow (C \uparrow C))^{\downarrow} = \neg(\neg(B \wedge B) \wedge \neg(C \wedge C)) \longleftrightarrow$
 338 $\neg\neg(B \vee C)$.

339 For the equivalence $\neg(\neg B \wedge \neg C) \longleftrightarrow \neg\neg(B \vee C)$: from left to right, if $\neg(B \vee C)$, then
 340 $\neg B$ and $\neg C$, so we have a contradiction with $\neg(\neg B \wedge \neg C)$; from right to left, if $\neg B \wedge \neg C$,
 341 then $\neg B$ and so from $B \vee C$ we derive C , contradiction, so we derive $\neg(B \vee C)$, but this
 342 contradicts $\neg\neg(B \vee C)$, so we conclude that $\neg(\neg B \wedge \neg C)$

3:16 Proof terms for generalized natural deduction

343 ■ $A = B \wedge C$. Then $(B \wedge C)^{\uparrow\downarrow} = ((B \uparrow C) \uparrow (B \uparrow C))^{\downarrow} = \neg(\neg(B \wedge C) \wedge \neg(B \wedge C)) \longleftrightarrow$
 344 $\neg\neg(B \wedge C)$.

345 ■ $A = B \rightarrow C$. Then $(B \rightarrow C)^{\uparrow\downarrow} = (B \uparrow (C \uparrow C))^{\downarrow} = \neg(B \wedge \neg(C \wedge C)) \longleftrightarrow \neg\neg(B \rightarrow C)$.
 346 For the equivalence $\neg(B \wedge \neg C) \longleftrightarrow \neg\neg(B \rightarrow C)$: From left to right, assume $\neg(B \rightarrow C)$;
 347 if C , then $B \rightarrow C$, so from $\neg(B \rightarrow C)$ we get $\neg C$; then if B we also have $B \wedge \neg C$,
 348 contradicting $\neg(B \wedge \neg C)$, so we have $\neg B$; but from $\neg B$ we get $B \rightarrow C$. Contradiction,
 349 so we conclude $\neg\neg(B \rightarrow C)$. From right to left: Assume $B \wedge \neg C$. Then $B \rightarrow C$ implies
 350 C , contradiction, so $\neg(B \rightarrow C)$, contradicting $\neg\neg(B \rightarrow C)$, so we conclude $\neg(B \wedge \neg C)$.
 351 ◀

► **Corollary 21.** For A a proposition in intuitionistic proposition logic,

$$\vdash_i \neg\neg A \quad \longleftrightarrow \quad \vdash_{\uparrow} A^{\uparrow}.$$

352 **Proof.** If $\vdash_i \neg\neg A$, then $\vdash_{\uparrow} \dot{\neg}\dot{\neg}A^{\uparrow}$ by Proposition 17, and so $\vdash_{\uparrow} \dot{\neg}\dot{\neg}A^{\uparrow}$ by Lemma 16(1).
 353 If $\vdash_{\uparrow} A^{\uparrow}$, then $\vdash_i A^{\uparrow\downarrow}$ by Proposition 19, so $\vdash_i \neg\neg A$ by Proposition 20. ◀

3 Convertibilities and conversion

354

355 The notion of *detour convertibility* has already been described in [7]: an introduction of Φ
 356 immediately followed by an elimination of Φ . (In [7] it was called *direct cut* but – although
 357 the literature is not completely consistent on this point – the notion of cut is usually reserved
 358 for sequent calculus and for natural deduction one uses the terminology of convertibility.) In
 359 such case there is (referring back to the truth table, see Definition 1) at least one k for which
 360 $a_k \neq b_k$. In case $a_k = 0, b_k = 1$, we have a sub-derivation Σ of $\vdash A_k$ and a sub-derivation Θ of
 361 $A_k \vdash D$ and we can plug Σ on top of Θ to obtain a derivation of $\vdash D$. In case $a_k = 1, b_k = 0$,
 362 we have a sub-derivation Σ of $A_k \vdash \Phi$ and a sub-derivation Θ of $\vdash A_k$ and we can plug Θ on
 363 top of Σ to obtain a derivation of $\vdash \Phi$. This is then used as a hypothesis for the elimination
 364 rule (that remains in this case) instead of the original one that was a consequence of the
 365 introduction rule (that now disappears).

366 In general there are more k for which $a_k \neq b_k$, so the general detour conversion procedure
 367 is non-deterministic. We view this non-determinism as a natural feature in natural deduction;
 368 the fact that for some connectives (or combination of connectives), detour conversion is
 369 deterministic is an “emerging” property. We will show examples of the non-determinism of
 370 detour conversion later.

371 The introduction of a formula Φ immediately followed by an elimination of Φ we will call
 372 a *detour convertibility*. In general in between the introduction rule for Φ and the elimination
 373 rule for Φ , there may be other auxiliary rules, so occasionally we may have to first permute
 374 the elimination rule with these auxiliary rules to obtain a detour convertibility that can
 375 be reduced away. So, we will also define the notion of *permutation convertibility* and of
 376 *permutation conversion*.

► **Definition 22.** Let c be a connective of arity n , with an elimination rule and an intuitionistic
 introduction rule derived from the truth table, as in Definition 1. So suppose we have the
 following rules in the truth table t_c .

p_1	...	p_n	$c(p_1, \dots, p_n)$
a_1	...	a_n	0
b_1	...	b_n	1

377 A *detour convertibility* in a derivation is a pattern of the following form, where $\Phi =$
 378 $c(A_1, \dots, A_n)$.

$$\frac{\dots \frac{\boxed{\Sigma_j}}{\Gamma \vdash A_j} \dots \dots \frac{\boxed{\Sigma_i}}{\Gamma, A_i \vdash \Phi} \dots}{\Gamma \vdash \Phi} \text{ in} \quad \frac{\dots \frac{\boxed{\Pi_k}}{\Gamma \vdash A_k} \dots \dots \frac{\boxed{\Pi_\ell}}{\Gamma, A_\ell \vdash D} \dots}{\Gamma \vdash D} \text{ el}$$

379 ■ Here, in is an arbitrary introduction rule. In this rule, A_j ranges over all propositions
 380 where $b_j = 1$; A_i ranges over all propositions where $b_i = 0$,

381 ■ Here, el is an arbitrary elimination rule. In this rule, A_k ranges over all propositions
 382 where $a_k = 1$; A_ℓ over all propositions where $a_\ell = 0$,

383 A *detour conversion* is defined by replacing the derivation pattern above by

1. If $\ell = j$ for some ℓ, j (that is: $A_\ell = A_j$):

$$\frac{\begin{array}{c} \vdots \boxed{\Sigma_j} \\ \Gamma \vdash A_j \end{array} \dots \begin{array}{c} \vdots \boxed{\Sigma_j} \\ \Gamma \vdash A_j \end{array}}{\Gamma \vdash D} \frac{\vdots \boxed{\Pi_\ell}}{\Gamma \vdash D} \text{ el}$$

2. If $k = i$ for some k, i (that is: $A_k = A_i$):

$$\frac{\begin{array}{c} \vdots \boxed{\Pi_k} \\ \Gamma \vdash A_i \end{array} \dots \begin{array}{c} \vdots \boxed{\Pi_k} \\ \Gamma \vdash A_i \end{array}}{\Gamma \vdash \Phi} \frac{\vdots \boxed{\Sigma_i}}{\Gamma \vdash \Phi} \dots \frac{\vdots \boxed{\Pi_k}}{\Gamma \vdash A_k} \dots \dots \frac{\vdots \boxed{\Pi_\ell}}{\Gamma, A_\ell \vdash D} \dots}{\Gamma \vdash D} \text{ el}$$

384

385 There may be several choices for the i and j in the previous definition, so detour elimination
 386 is non-deterministic in general. We give an example of most to illustrate this. For simplicity,
 387 we use the optimized rules.

► **Example 23.** Consider the following detour convertibility for most.

$$\frac{\frac{\begin{array}{c} \vdots \boxed{\Sigma_1} \\ \Gamma \vdash A \end{array} \quad \begin{array}{c} \vdots \boxed{\Sigma_2} \\ \Gamma \vdash B \end{array}}{\Gamma \vdash \text{most}(A, B, C)} \text{ most-in}_1 \quad \frac{\begin{array}{c} \vdots \boxed{\Pi_1} \\ \Gamma, A \vdash D \end{array} \quad \begin{array}{c} \vdots \boxed{\Pi_2} \\ \Gamma, B \vdash D \end{array}}{\Gamma \vdash D} \text{ most-el}_1$$

388 Here we can reduce to either one of the following derivations of $\Gamma \vdash D$, which shows that
 389 the detour conversion process is not Church-Rosser. (Of course, one could fix a choice, e.g.
 390 always take the first possible detour convertibility from the left, but that would be completely
 391 arbitrary.)

$$\begin{array}{c} \vdots \boxed{\Sigma_1} \\ \Gamma \vdash A \end{array} \dots \begin{array}{c} \vdots \boxed{\Sigma_1} \\ \Gamma \vdash A \end{array} \quad \frac{\vdots \boxed{\Pi_1}}{\Gamma \vdash D}$$

$$\begin{array}{c} \vdots \boxed{\Sigma_2} \\ \Gamma \vdash B \end{array} \dots \begin{array}{c} \vdots \boxed{\Sigma_2} \\ \Gamma \vdash B \end{array} \quad \frac{\vdots \boxed{\Pi_2}}{\Gamma \vdash D}$$

3:18 Proof terms for generalized natural deduction

A more concrete example is the following.

$$\frac{\frac{A \wedge B \vdash A \wedge B}{A \wedge B \vdash A} \wedge\text{-ell} \quad \frac{A \wedge B \vdash A \wedge B}{A \wedge B \vdash B} \wedge\text{-elr} \quad \frac{A \vdash A}{A \vdash A \vee B} \vee\text{-inl} \quad \frac{B \vdash B}{B \vdash A \vee B} \vee\text{-inr}}{A \wedge B \vdash \text{most}(A, B, C)} \text{most-in}_1 \quad \frac{A \wedge B \vdash \text{most}(A, B, C) \quad \frac{A \vdash A}{A \vdash A \vee B} \vee\text{-inl} \quad \frac{B \vdash B}{B \vdash A \vee B} \vee\text{-inr}}{A \wedge B \vdash A \vee B} \text{most-el}_1$$

392 This derivation can either be reduced to a derivation of $A \wedge B \vdash A \vee B$ via $A \wedge B \vdash A$ or via
393 $A \wedge B \vdash B$.

394 It can happen that the introduction of a formula $\Phi = c(A_1, \dots, A_n)$ is not followed
395 directly by an elimination for c , but first by other elimination rules, where Φ acts as a minor
396 premise. In that way, a detour convertibility can be “blocked” by other elimination rules. So,
397 apart from the detour conversion elimination arising from an introduction rule immediately
398 followed by an elimination, we have a notion of “hidden” or *permutation convertibility*, where
399 we want to permute one elimination rule over another.

► **Example 24.**

$$\frac{\frac{\Gamma \vdash A \vee B \quad \frac{\Gamma, A, C \vdash C \rightarrow D}{\Gamma, A \vdash C \rightarrow D} \rightarrow\text{-in}_a}{\Gamma \vdash C \rightarrow D} \vee\text{-el} \quad \frac{\Gamma, B \vdash C \rightarrow D}{\Gamma \vdash C \rightarrow D} \vee\text{-el}}{\Gamma \vdash D} \rightarrow\text{-el}$$

400 In this derivation, the detour convertibility arising from $\rightarrow\text{-in}_a$ followed by $\rightarrow\text{-el}$ is blocked
401 by the $\vee\text{-el}$ rule where the major premise of the $\rightarrow\text{-el}$ rule is a minor premise. This is a
402 *permutation convertibility*, which can be contracted by permuting the $\rightarrow\text{-el}$ rule over the $\vee\text{-el}$
403 rule.

► **Definition 25.** Let c and c' be connectives of arity n and n' , with elimination rules r
and r' respectively, both derived from the truth table. A *permutation convertibility* in a
derivation is a pattern of the following form, where $\Phi = c(B_1, \dots, B_n)$, $\Psi = c'(A_1, \dots, A_{n'})$.

$$\frac{\frac{\Gamma \vdash \Psi \dots \Gamma \vdash A_j \quad \dots \quad \dots \quad \Gamma, A_i \vdash \Phi \quad \dots}{\Gamma \vdash \Phi} \text{el}_{r'} \quad \frac{\dots \quad \Gamma \vdash B_k \quad \dots \quad \dots \quad \Gamma, B_\ell \vdash D \quad \dots}{\Gamma \vdash D} \text{el}_r}{\Gamma \vdash D} \text{el}_{r'}$$

404 ■ A_j ranges over all propositions that have a 1 in the truth table of c' ; A_i ranges over all
405 propositions that have a 0,

406 ■ B_k ranges over all propositions that have a 1 in the truth table of c ; B_ℓ ranges over all
407 propositions that have a 0.

408 The *permutation conversion* is defined by replacing the derivation pattern above by

$$\frac{\frac{\Gamma \vdash \Psi \dots \Gamma \vdash A_j \quad \dots \quad \dots \quad \frac{\Gamma, A_i \vdash \Phi \quad \dots \quad \Gamma, A_i \vdash B_k \quad \dots \quad \dots \quad \Gamma, A_i, B_\ell \vdash D \quad \dots}{\Gamma, A_i \vdash D} \text{el}_r}{\Gamma \vdash D} \text{el}_{r'}}$$

409 This gives rise to copying of sub-derivations: for every A_i we copy the sub-derivations
410 Π_1, \dots, Π_n .

411 NB. Due to weakening, $\boxed{\Pi_k}$ is also a derivation of $\Gamma, A_i \vdash B_k$ and $\boxed{\Pi_\ell}$ is also a derivation
 412 of $\Gamma, A_i, B_\ell \vdash D$.

► **Example 26.** If we reduce the permutation convertibility in Example 24, we obtain the following derivation.

$$\frac{\frac{\frac{\Gamma, A, C \vdash C \rightarrow D}{\Gamma, A \vdash C \rightarrow D} \rightarrow\text{-in}_a \quad \Gamma, A \vdash C}{\Gamma, A \vdash D} \rightarrow\text{-el} \quad \frac{\Gamma, B \vdash C \rightarrow D \quad \Gamma, B \vdash C}{\Gamma, B \vdash D} \rightarrow\text{-el}}{\Gamma \vdash A \vee B \quad \Gamma \vdash D} \vee\text{-el}$$

413 4 The Curry-Howard isomorphism

414 We now define typed proof-terms for derivations, which enables the study of “proofs as
 415 terms” and emphasizes the computational interpretation of proofs, as detour conversion
 416 and permutation conversion will correspond to reductions on these proof-terms. For each
 417 connective c we give a general definition of proof-terms for the full set of derivation rules for c ,
 418 as they have been derived from the truth table. This amounts to a system $\lambda^{\mathcal{C}}$, parametrized
 419 by a set of connectives \mathcal{C} . Then, to clarify the approach, we show how this works out on a
 420 number of examples.

421 Often, we don’t want to consider the full rules for a connective c , but only the optimized
 422 rules, following Lemmas 9 and 12. For these optimized rules, there is also a straightforward
 423 definition of proof-terms and of the reduction relation associated with (detour, permutation)
 424 conversion. In the next Section 5 we show in detail how Lemmas 9 and 12 can be extended
 425 to terms and reductions: the proof-terms for the optimized rules can be defined in terms
 426 of our original calculus $\lambda^{\mathcal{C}}$, and the reduction rules for the optimized proof terms are an
 427 instance of reductions in the original calculus (often multi-step).

428 ► **Definition 27.** Given a logic with intuitionistic derivation rules, as derived from truth
 429 tables for a set of connectives \mathcal{C} , as in Definition 1, we now define the typed λ -calculus $\lambda^{\mathcal{C}}$.
 430 The system $\lambda^{\mathcal{C}}$ has judgments $\Gamma \vdash t : A$, where A is a formula, Γ is a set of declarations
 431 $\{x_1 : A_1, \dots, x_m : A_m\}$, where the A_i are formulas and the x_i are term-variables such that
 432 every x_i occurs at most once in Γ , and t is a *proof-term*.

433 Let $c \in \mathcal{C}$ be a connective of arity n , which has 2^n rules (introduction plus elimination
 434 rules). For each rule r we have a term: an *introduction term*, $\{\bar{p} ; \bar{Q}\}_r$, if r is an introduction
 435 rule, or an *elimination term*, $t \cdot_r [\bar{p} ; \bar{Q}]$, if r is an elimination rule. Here, t is again a term, \bar{p}
 436 is a finite sequence of terms and \bar{Q} is a finite sequence of *abstracted terms* $\lambda x : A.q$, where x
 437 is a term-variable, A is a proposition and q is a term. So the abstract syntax for proof-terms,
 438 **Term**, is as follows.

$$t ::= x \mid \{\bar{t} ; \overline{\lambda x : A.t}\}_r \mid t \cdot_r [\bar{t} ; \overline{\lambda x : A.t}]$$

439 where x ranges over variables and r ranges over the rules of all the connectives.

3:20 Proof terms for generalized natural deduction

The terms are *typed* using the following derivation rules.

$$\begin{array}{c}
 \frac{}{\Gamma \vdash x_i : A_i} \text{ if } x_i : A_i \in \Gamma \\
 \frac{\dots \Gamma \vdash p_j : A_j \dots \quad \dots \Gamma, y_i : A_i \vdash q_i : \Phi \dots}{\Gamma \vdash \{\bar{p}; \overline{\lambda y : A.q}\}_r : \Phi} \text{ in} \\
 \frac{\Gamma \vdash t : \Phi \quad \dots \Gamma \vdash p_k : A_k \dots \quad \dots \Gamma, y_\ell : A_\ell \vdash q_\ell : D}{\Gamma \vdash t \cdot_r [\bar{p}; \overline{\lambda y : A.q}] : D} \text{ el}
 \end{array}$$

440 Here, \bar{p} is the sequence of terms $p_1, \dots, p_{m'}$ for all the 1-entries in the truth table, and
 441 $\overline{\lambda y : A.q}$ is the sequence of terms $\lambda y_1 : A_1.q_1, \dots, \lambda y_m : A_m.q_m$ for all the 0-entries in the
 442 truth table.

443 ► **Convention 28.** We view the λ -abstracted variables as being *typed* so we write $\overline{\lambda y : A.q}$
 444 and $\lambda y_1 : A_1.q_1, \dots, \lambda y_m : A_m.q_m$. However, these types clutter up the syntax considerably,
 445 so in practice we will almost always leave the types implicit. In case we want to stress that a
 446 variable has a certain type, or in case type information enhances the understanding, we will
 447 write the type as a superscript, so $\lambda x^A.p$ in stead of $\lambda x : A.p$.

448 We will sometimes leave the rule r that the elimination or introduction term corresponds
 449 to implicit, or we will just number the terms or introduce special names for them without
 450 explicit reference to the rule. It should be clear that every line in the truth table for the
 451 connective gives rise to one rule, which again gives rise to one term-constructor, which is
 452 either an elimination or an introduction term-constructor.

453 There are term reduction rules that correspond to detour conversion.

454 ► **Definition 29.** Given a detour convertibility as defined in Definition 22, we add reduction
 455 rules for the associated terms as follows.

- For the $\ell = j$ case, that is, $y_\ell : A_\ell$ and $p_j : A_j$ with $A_\ell = A_j$:

$$\{\bar{p}, p_j; \overline{\lambda x.q}\} \cdot [\bar{s}; \overline{\lambda y.r, \lambda y_\ell.r_\ell}] \longrightarrow_a r_\ell[y_\ell := p_j]$$

- For the $k = i$ case, that is, $s_k : A_k$ and $x_i : A_i$ with $A_k = A_i$:

$$\{\bar{p}; \overline{\lambda x.q, \lambda x_i.q_i}\} \cdot [\bar{s}, s_k; \overline{\lambda y.r}] \longrightarrow_a q_i[x_i := s_k] \cdot [\bar{s}, \overline{\lambda y.r}]$$

456 For simplicity of presentation we write the “matching cases” in Definition 22 as last term
 457 of the sequence. So when writing \bar{p}, p_j , this should be understood as a sequence of terms
 458 $p_1, \dots, p_j, \dots, p_{m'}$, where we have singled out the p_j that matches the r_ℓ in $\overline{\lambda y.r, \lambda y_\ell.r_\ell}$.
 459 Similarly for \bar{s}, s_k and $\overline{\lambda x.q, \lambda x_i.q_i}$.

460 It is important to note that there is always (at least one) “matching case”, because
 461 introduction rules and elimination rules comes from different lines in the truth table.

462 The reduction is extended in the straightforward way to sub-terms, by defining it as a
 463 congruence with respect to the term constructions.

464 This Definition gives a reduction rule, and possibly more than one, for every combination
 465 of an elimination and an introduction. For an n -ary connective, there are 2^n rules in the
 466 truth table, and therefore 2^n term-constructors (introduction plus elimination constructors).
 467 We now give the examples of the proof-terms for \vee and \wedge in full. In the rules we will always
 468 omit the context Γ .

► **Example 30.** The rules for disjunction are as follows.

$\frac{\vdash t : A \vee B \quad x : A \vdash p : D \quad y : B \vdash q : D}{\vdash t \cdot^\vee [; \lambda x.p, \lambda y.q] : D}$	$\frac{z : A \vdash r : A \vee B \quad \vdash b : B}{\vdash \{b ; \lambda z.r\}_1^\vee : A \vee B}$
$\frac{\vdash a : A \quad z : B \vdash r : A \vee B}{\vdash \{a ; \lambda z.r\}_2^\vee : A \vee B}$	$\frac{\vdash a : A \quad \vdash b : B}{\vdash \{a, b ; \}_3^\vee : A \vee B}$

469 We could have followed our earlier introduced naming convention and index the operators
 470 with the line of the truth table they arise from. Then we would write $\{b ; \lambda z.r\}_{01}^\vee$ for
 471 $\{b ; \lambda z.r\}_1^\vee$, $\{a ; \lambda z.r\}_{10}^\vee$ for $\{a ; \lambda z.r\}_2^\vee$ and $\{a, b ; \}_{11}^\vee$ $\{a, b ; \}_3^\vee$. This easily clutters up
 472 notation, so we don't pursue that.

473 The reduction rules are

474 $\{b ; \lambda z.r\}_1^\vee \cdot^\vee [; \lambda x.p, \lambda y.q] \longrightarrow_a q[y := b]$
 475 $\{a ; \lambda z.r\}_2^\vee \cdot^\vee [; \lambda x.p, \lambda y.q] \longrightarrow_a p[x := a]$
 476 $\{a, b ; \}_3^\vee \cdot^\vee [; \lambda x.p, \lambda y.q] \longrightarrow_a p[x := a]$
 477 $\{a, b ; \}_3^\vee \cdot^\vee [; \lambda x.p, \lambda y.q] \longrightarrow_a q[y := b]$

478 From the last two cases, we see that the Church-Rosser property (confluence) is lost.
 The rules for conjunction are as follows.

$\frac{\vdash t : A \wedge B \quad x : A \vdash p : D \quad y : B \vdash q : D}{\vdash t \cdot^\wedge [; \lambda x.p, \lambda y.q] : D}$	$\frac{\vdash t : A \wedge B \quad \vdash a : A \quad y : B \vdash q : D}{\vdash t \cdot^\wedge [a ; \lambda y.q] : D}$
$\frac{\vdash t : A \wedge B \quad x : A \vdash p : D \quad \vdash b : B}{\vdash t \cdot^\wedge [b ; \lambda x.p] : D}$	$\frac{\vdash a : A \quad \vdash b : B}{\vdash \{a, b ; \}^\wedge : A \wedge B}$

479 The reduction rules are

480 $\{a, b ; \}^\wedge \cdot^\wedge [; \lambda x.p, \lambda y.q] \longrightarrow_a p[x := a]$
 481 $\{a, b ; \}^\wedge \cdot^\wedge [; \lambda x.p, \lambda y.q] \longrightarrow_a q[y := b]$
 482 $\{a, b ; \}^\wedge \cdot^\wedge [a' ; \lambda y.q] \longrightarrow_a q[y := b]$
 483 $\{a, b ; \}^\wedge \cdot^\wedge [b' ; \lambda x.p] \longrightarrow_a p[x := a]$

484 From the first two cases, we see that the Church-Rosser property (confluence) is lost.

485 In Example 39 we will show how we can define proof-terms for the optimized rules for \wedge
 486 in terms of the proof-terms for the full rules, while preserving reduction.

487 In the reduction for the terms for \vee and \wedge , an elimination is always removed at each step.
 488 The situation gets more interesting with implication.

► **Example 31.** The rules for implication are as follows.

$\frac{x : A \vdash p : A \rightarrow B \quad y : B \vdash q : A \rightarrow B}{\vdash \{ ; \lambda x.p, \lambda y.q \}_1^\rightarrow : A \rightarrow B}$	$\frac{x : A \vdash p : A \rightarrow B \quad \vdash b : B}{\vdash \{b ; \lambda x.p\}_2^\rightarrow : A \rightarrow B}$
$\frac{\vdash t : A \rightarrow B \quad \vdash a : A \quad z : B \vdash r : D}{\vdash t \cdot^\rightarrow [a ; \lambda z.r] : D}$	$\frac{\vdash a : A \quad \vdash b : B}{\vdash \{a, b ; \}_3^\rightarrow : A \rightarrow B}$

3:22 Proof terms for generalized natural deduction

489 The reduction rules are

$$\begin{aligned}
 490 \quad & \{ ; \lambda x.p, \lambda y.q \}_1^{\rightarrow} \cdot^{\rightarrow} [a ; \lambda z.r] \longrightarrow_a p[x := a] \cdot^{\rightarrow} [a ; \lambda z.r] \\
 491 \quad & \{ b ; \lambda x.p \}_2^{\rightarrow} \cdot^{\rightarrow} [a ; \lambda z.r] \longrightarrow_a r[z := b] \\
 492 \quad & \{ b ; \lambda x.p \}_2^{\rightarrow} \cdot^{\rightarrow} [a ; \lambda z.r] \longrightarrow_a p[x := a] \cdot^{\rightarrow} [a ; \lambda z.r] \\
 493 \quad & \{ a', b ; \}_3^{\rightarrow} \cdot^{\rightarrow} [a ; \lambda z.r] \longrightarrow_a r[z := b]
 \end{aligned}$$

494 From the second and third case, we can see that Church-Rosser is lost. In the first and the
 495 third case, we see that the elimination remains.

496 In Example 41 we will show how we can define proof-terms for the optimized rules for \rightarrow
 497 in terms of the proof-terms for the full rules, while preserving reduction. In Definition 48 we
 498 will define the standard rules for \rightarrow .

499 We now extend the reduction on proof-terms to also capture the permutation conversions
 500 of Definition 25. This gives rise to two elimination constructs permuting with each other.

501 **► Definition 32.** Given a permutation convertibility as defined in Definition 25, we add
 502 reduction rules for the associated terms as follows.

$$503 \quad (t \cdot [\bar{p} ; \overline{\lambda x.q}] \cdot [\bar{s} ; \overline{\lambda y.r}]) \longrightarrow_b t \cdot [\bar{p} ; \overline{\lambda x.(q \cdot [\bar{s} ; \overline{\lambda y.r}])}]$$

504 Here, the notation $\overline{\lambda x.(q \cdot [\bar{s} ; \overline{\lambda y.r}])}$ should be understood as a sequence $\lambda x_1.q_1, \dots, \lambda x_m.q_m$
 505 where each q_j is replaced by $q_j \cdot [\bar{s} ; \overline{\lambda y.r}]$.

506 The reduction is extended in the straightforward way to sub-terms, by defining it as a
 507 congruence with respect to the term constructions.

508 **► Notation 33.** We omit brackets by letting the application operator $- \cdot -$ associate to the
 509 left, so $t \cdot [\bar{p} ; \overline{\lambda x.q}] \cdot [\bar{s} ; \overline{\lambda y.r}]$ denotes $(t \cdot [\bar{p} ; \overline{\lambda x.q}]) \cdot [\bar{s} ; \overline{\lambda y.r}]$. We will also omit the brackets
 510 in $\overline{\lambda x.(q \cdot [\bar{s} ; \overline{\lambda y.r}])}$, because no ambiguity can arise here.

511 We treat the well-known example from intuitionistic logic of the \vee -elimination, where a
 512 permutation convertibility can occur. See also Example 24.

► Example 34.

$$\frac{\frac{\frac{\vdash t : A \vee B \quad x : A \vdash p : C \rightarrow D \quad y : B \vdash q : C \rightarrow D}{\vdash t \cdot^{\vee} [; \lambda x.p, \lambda y.q] : C \rightarrow D}}{\vdash t \cdot^{\vee} [; \lambda x.p, \lambda y.q] \cdot^{\rightarrow} [c ; \lambda z.r] : E}}{\vdash c : C \quad z : D \vdash r : E}$$

513 We observe two consecutive elimination rules, where a potential detour convertibility, arising
 514 e.g. when q is an introduction term, is blocked by the \vee -elimination.

The term reduces as follows

$$t \cdot^{\vee} [; \lambda x.p, \lambda y.q] \cdot^{\rightarrow} [c ; \lambda z.r] \longrightarrow_b t \cdot^{\vee} [; \lambda x.p \cdot^{\rightarrow} [c ; \lambda z.r], \lambda y.q \cdot^{\rightarrow} [c ; \lambda z.r]]$$

515 We can now easily define the terms in normal-form under the combined reduction \longrightarrow_{ab} .
 516 The proof is straightforward and comes from the fact that an introduction followed by an
 517 elimination is always a redex. (There is always a “matching case” in Definition 29.)

518 **► Lemma 35.** *The set of terms in normal form of IPC_C , NF is characterized by the following*
 519 *inductive definition.*

520 **■** $x \in \text{NF}$ for every variable x ,

- 521 ■ $\{\overline{p}; \overline{\lambda y.q}\} \in \text{NF}$ if all p_i and q_j are in NF,
 522 ■ $x \cdot \overline{p}; \overline{\lambda y.q} \in \text{NF}$ if all p_i and q_j are in NF and x is a variable.

523 ► **Remark.** In [23], yet another notion of convertibility is defined, called *simplification*
 524 *convertibility*. This is a situation where the assumption is unused in an introduction or
 525 elimination rule and the rule can be removed all together. Adding these rules is not necessary
 526 for the sub-formula property, so we don't introduce it here. On the term level, an elimination
 527 of simplification convertibilities would amount to the following reduction rules.

$$528 \quad t \cdot \overline{p}; \overline{\lambda x.q} \longrightarrow q_i \quad \text{if } x_i \notin \text{FV}(q_i)$$

$$529 \quad \{\overline{p}; \overline{\lambda x.q}\} \longrightarrow q_i \quad \text{if } x_i \notin \text{FV}(q_i)$$

5 Extending the Curry-Howard isomorphism to definable rules

531 The optimizations for the logical rules, as given in Lemmas 9 and 12 can be extended to
 532 the proof terms and also to convertibilities and conversions. This gives us the possibility
 533 to capture questions related to normalization by looking at normalization for terms in the
 534 original calculus λ^c . We will now describe the terms for the optimized rules in detail.

535 ► **Definition 36.** For each optimization step in Lemmas 9 and 12 we give the canonical term
 536 for the optimized rule and its translation in terms of λ^c of Definition 27.

537 We first treat the two optimizations arising from Lemma 9, and then the optimization
 538 arising from Lemma 12.

- Given two rules

$$\frac{\vdash p_1 : A_1 \dots \vdash p_n : A_n \quad x_1 : B_1 \vdash q_1 : \Phi \dots x_m : B_m \vdash q_m : \Phi \quad z : A \vdash s : \Phi}{\vdash \{\overline{p}; \overline{\lambda x.q, \lambda z.s}\}_r : \Phi} \text{in}_r$$

$$\frac{\vdash p_1 : A_1 \dots \vdash p_n : A_n \quad \vdash a : A \quad x_1 : B_1 \vdash q_1 : \Phi \dots x_m : B_m \vdash q_m : \Phi}{\vdash \{\overline{p, a}; \overline{\lambda x.q}\}_{r'} : \Phi} \text{in}_{r'}$$

we have the following term for the optimized introduction rule

$$\frac{\vdash p_1 : A_1 \dots \vdash p_n : A_n \quad x_1 : B_1 \vdash q_1 : \Phi \dots x_m : B_m \vdash q_m : \Phi}{\vdash \{\overline{p}; \lambda x.q, \lambda z.\{\overline{p, z}; \overline{\lambda x.q}\}_{r'}\}_r : \Phi} \text{in}_{r, r'}^{\text{opt}}$$

539 We define the term $\{\overline{p}; \overline{\lambda x.q}\}_{r, r'}^{\circ}$ as $\{\overline{p}; \overline{\lambda x.q, \lambda z.\{\overline{p, z}; \overline{\lambda x.q}\}_{r'}}\}_r$

- Given two rules

$$\frac{\vdash t : \Phi \quad \vdash p_1 : A_1 \dots \vdash p_n : A_n \quad x_1 : B_1 \vdash q_1 : D \dots x_m : B_m \vdash q_m : D \quad z : A \vdash s : D}{\vdash t \cdot_r \overline{p}; \overline{\lambda x.q, \lambda z.s} : D} \text{el}_r$$

$$\frac{\vdash t : \Phi \quad \vdash p_1 : A_1 \dots \vdash p_n : A_n \quad \vdash a : A \quad x_1 : B_1 \vdash q_1 : D \dots x_m : B_m \vdash q_m : D}{\vdash t \cdot_{r'} \overline{p, a}; \overline{\lambda x.q} : D} \text{el}_{r'}$$

we have the following term for the optimized elimination rule

$$\frac{\vdash t : \Phi \quad \vdash p_1 : A_1 \dots \vdash p_n : A_n \quad x_1 : B_1 \vdash q_1 : D \dots x_m : B_m \vdash q_m : D}{\vdash t \cdot_r \overline{p}; \lambda x.q, \lambda z.t \cdot_{r'} \overline{p, z}; \overline{\lambda x.q}} \text{el}_{r, r'}^{\text{opt}}$$

540 We define term $t \odot_{r, r'} \overline{p}; \overline{\lambda x.q}$ as $t \cdot_r \overline{p}; \overline{\lambda x.q, \lambda z.t \cdot_{r'} \overline{p, z}; \overline{\lambda x.q}}$

■ Given the rule

$$\frac{\vdash t : \Phi \quad \vdash p_1 : A_1 \dots \vdash p_n : A_n \quad z : A \vdash s : D}{\vdash t \cdot_r [\bar{p} ; \lambda z.s] : D} \text{el}_r$$

we have the following term for the optimized elimination rule

$$\frac{\vdash t : \Phi \quad \vdash p_1 : A_1 \dots \vdash p_n : A_n}{\vdash t \cdot_r [\bar{p} ; \lambda z.z] : A} \text{el}_r^{\text{opt}}$$

541 We define the term $t \square_r [\bar{p}]$ as $t \cdot_r [\bar{p} ; \lambda z.z]$

542 There is a canonical way in which the notions of detour convertibility and detour conversion
 543 extend to the optimized rules: the same rules as in Definition 29 apply. In case of a term of
 544 the form $\{\dots ; \dots\} \cdot [\dots ; \dots]$, a reduction is always possible, also in the case of optimized
 545 rules. For the permutation convertibilities, the situation is similar: the same rules as in
 546 Definition 32 apply.

547 ► **Definition 37.** We define reduction on the optimized terms as follows. Let \odot be any $\cdot_{r''}$
 548 or $\odot_{r'', r'''}$ for some r'', r''' . (For the notation, we refer to Definition 29.)

549 For the $\ell = j$ case:

$$550 \{\bar{p}, p_j ; \overline{\lambda x.q}\}_{r, r'}^\circ \odot [\bar{s} ; \overline{\lambda y.u}, \overline{\lambda y_\ell.u_\ell}] \longrightarrow_a u_\ell[y_\ell := p_j]$$

551 For the $k = i$ case:

$$552 \{\bar{p} ; \overline{\lambda x.q}, \overline{\lambda x_i.q_i}\}_{r, r'}^\circ \odot [\bar{s}, s_k ; \overline{\lambda y.u}] \longrightarrow_a q_i[x_i := s_k] \odot [\bar{s}, s_k ; \overline{\lambda y.u}]$$

553 For the $k = i$ case:

$$554 \{\bar{s} ; \overline{\lambda x.q}\}_{r, r'}^\circ \square_r [\bar{p}] \longrightarrow_a q_i[x_i := p_k] \square_r [\bar{p}]$$

555 Special case:

$$556 \{\bar{s}, s_j ; \overline{\lambda x.q}\}_{r, r'}^\circ \square_r [\bar{p}] \longrightarrow_a s_j$$

557 The last special case is when $\{\bar{s}, s_j ; \overline{\lambda x.q}\}_{r, r'}^\circ \square_r [\bar{p}] : A$ and $s_j : A$. See the definition of
 558 $\{\bar{s}, s_j ; \overline{\lambda x.q}\}_{r, r'}^\circ \square_r [\bar{p}]$ as $\{\bar{s}, s_j ; \overline{\lambda x.q}\}_{r, r'}^\circ \cdot_r [\bar{p} ; \lambda z.z]$ in Definition 36; this is the case where
 559 s_j matches the “invisible” $\lambda z.z$.

560 We also extend the notions of permutation convertibility and permutation conversion
 561 from Definition 25 (see also Definition 32): we add reduction rules for the optimized terms
 562 as follows.

$$563 (t \ominus [\bar{p} ; \overline{\lambda x.q}]) \odot [\bar{s} ; \overline{\lambda y.u}] \longrightarrow_b t \ominus [\bar{p} ; \overline{\lambda x.(q \odot [\bar{s} ; \overline{\lambda y.u}])}]$$

564 where \ominus is any $\cdot_{r''}$ or $\odot_{r'', r'''}$ and \odot is any $\cdot_{r''}$ or $\odot_{r'', r'''}$ or $\square_{r''}$.

► **Remark.** To clarify, we want to note explicitly that $t \square_r [\bar{p}] \cdot_{r'} [\bar{q} ; \overline{\lambda x.s}]$ does *not* reduce to
 $t \square_r [\bar{p}]$. In case we only have the optimized rules, it does not reduce at all. If we consider
 $t \square_r [\bar{p}]$ as a definition in the original calculus λ^C , we do have a reduction,

$$t \square_r [\bar{p}] \cdot_{r'} [\bar{q} ; \overline{\lambda x.s}] \longrightarrow_b t \cdot_r [\bar{p} ; \lambda z.z \cdot_{r'} [\bar{q} ; \overline{\lambda x.s}]]$$

565 but this uses a non-optimized elimination.

566 ► **Remark.** The process described in Definition 36, which is based on Lemmas 9 and 12
 567 can be iterated, as we have seen in earlier examples. A simple way to view the rules
 568 for an n -ary connective c as a pair (b, r) where b is 0 or 1 and r is a partial function
 569 $r : \{1, 2, \dots, n\} \rightarrow \{0, 1\}$. For a standard rule, derived from a line in the truth table of c ,

570 r is a total function. (If $r(i) = 1$, then A_i is a lemma in the rule and if $r(j) = 0$, then A_j is
 571 a case; if $b = 0$, we have an elimination rule, if $b = 1$ we have an introduction rule.) An
 572 optimized rule is a function r that is undefined for some elements of $\{1, \dots, n\}$.

573 For the first case of Definition 36, where $\{\dots; \dots\}_{r,r'}^\circ$ is defined in terms of $\{\dots; \dots\}_r$
 574 and $\{\dots; \dots\}_{r'}$, we have $r'' = r \cap r'$ for the optimized rule r'' . This is allowed in case $b = 1$
 575 for r and r' and r and r' differ for only one element.

576 For the second case of Definition 36, where $\dots \odot_{r,r'} [\dots; \dots]$ is defined in terms of
 577 $\dots \cdot_r [\dots; \dots]$ and $\dots \cdot_{r'} [\dots; \dots]$, we again have $r'' = r \cap r'$ for the optimized rule r'' . This
 578 is allowed in case $b = 0$ for r and r' and r and r' differ for only one element.

579 Optimization according to Lemma 12, the third case of Definition 36, corresponds with a
 580 (possibly partial) function r where $b = 0$ and $r(i) = 1$ for exactly one i .

581 With the definable optimized terms for elimination and introduction, we have a choice
 582 of taking these as defined terms, or taking them as primitives and removing the originals.
 583 Or even there is a third alternative of adding them as additional term constructions. After
 584 we have done some examples, we will, in Lemma 43, analyze the reduction behaviour of the
 585 newly defined terms in terms of the original ones.

586 Before that we state what the normal forms are of the optimized terms and the optimized
 587 reduction, extending Lemma 35. So in the following Lemma, we consider the situation where
 588 we have added optimized terms and reductions, while removing the original ones. The proof
 589 is straightforward, keeping in mind Remark 5 and the fact that with optimized terms, if an
 590 introduction is followed immediately by an elimination, then there is a “matching case” that
 591 allows us to reduce the term.

592 ► **Lemma 38.** *We simultaneously characterize NF^{opt} , the set of terms in normal form of*
 593 *$\text{IPC}_{\mathcal{C}}$ with optimized terms and reductions, and the set of neutral terms inductively as follows.*

- 594 ■ $x \in \text{NF}^{\text{opt}}$ and x is neutral, for every variable x ,
- 595 ■ $\{\bar{p}; \overline{\lambda y. q}\} \in \text{NF}^{\text{opt}}$ if all p_i and q_j are in NF^{opt} ,
- 596 ■ $x \odot [\bar{p}; \overline{\lambda y. q}] \in \text{NF}^{\text{opt}}$ if all p_i and q_j are in NF^{opt} and x is a variable; this term is neutral
 597 if $\odot = \square_r$ for some r .
- 598 ■ $t \square_r [\bar{s}] \odot [\bar{p}; \overline{\lambda y. q}] \in \text{NF}^{\text{opt}}$ if all s_k , p_i and q_j are in NF^{opt} and t is neutral; this term is
 599 neutral if $\odot = \square_{r'}$ for some r' .

What the Lemma says is that terms like

$$x \square_r [\bar{s}_1] \square_{r'} [\bar{s}_2] \square_{r''} \dots \odot [\bar{p}; \overline{\lambda y. q}]$$

600 are also normal forms, if $\bar{s}_1, \bar{s}_2, \dots, \bar{p}$ and \bar{q} are.

► **Example 39.** We continue Example 30 and look into the optimized rules for \wedge , as given
 in Definition 13. The introduction rule of Example 30 is the same as in Definition 13; the
 usual “pairing” construction is given by $\{a, b; \}^\wedge$. For elimination, we would like to have
 the following “projection” rules.

$$\frac{\vdash t : A \wedge B}{\vdash \pi_1 t : A} \quad \frac{\vdash t : A \wedge B}{\vdash \pi_2 t : B}$$

That is, we would like to define $\pi_1 t$ and $\pi_2 t$ in terms of the constructions of Example 30,
 with the expected reduction rules: $\pi_1 \{a, b; \}^\wedge \rightarrow_a a$ and $\pi_2 \{a, b; \}^\wedge \rightarrow_a b$. Definition
 36 gives the clue. Let’s consider the first projection, $\pi_1 t$. We have the following optimization
 of the \wedge -rules of Example 30.

$$\frac{\vdash t : A \wedge B \quad x : A \vdash p : D}{t \odot_a^\wedge [; \lambda x^A. p] : D}$$

3:26 Proof terms for generalized natural deduction

where $t \odot_a^\wedge [; \lambda x^A.p] := t \cdot \hat{\cdot}_1 [; \lambda x^A.p, \lambda z^B.t \cdot \hat{\cdot}_3 [z; \lambda x^A.p]]$. It is easily verified that we have the following reduction

$$\{a, b; \}^\wedge \odot_a^\wedge [; \lambda x^A.p] \longrightarrow_a p[x := a].$$

We have another optimization:

$$\frac{\vdash t : A \wedge B}{\vdash t \sqsupset_1^\wedge [;] : A}$$

601 where $t \sqsupset_1^\wedge [;] := t \odot_a^\wedge [; \lambda x^A.x]$.

All together we have

$$\pi_1 t := t \sqsupset_1^\wedge [;] = t \odot_a^\wedge [; \lambda x^A.x] = t \cdot \hat{\cdot}_1 [; \lambda x^A.x, \lambda z^B.t \cdot \hat{\cdot}_3 [z; \lambda x^A.x]]$$

602 which has the following reductions.

$$603 \quad \pi_1 \{a, b; \}^\wedge = \{a, b; \}^\wedge \cdot \hat{\cdot}_1 [; \lambda x^A.x, \lambda z^B.\{a, b; \}^\wedge \cdot \hat{\cdot}_3 [z; \lambda x^A.x]]$$

$$604 \quad \longrightarrow_a a$$

$$605 \quad \pi_1 \{a, b; \}^\wedge = \{a, b; \}^\wedge \cdot \hat{\cdot}_1 [; \lambda x^A.x, \lambda z^B.\{a, b; \}^\wedge \cdot \hat{\cdot}_3 [z; \lambda x^A.x]]$$

$$606 \quad \longrightarrow_a \{a, b; \}^\wedge \cdot \hat{\cdot}_3 [b; \lambda x^A.x]$$

$$607 \quad \longrightarrow_a a$$

608 Similarly, we define $\pi_2 t := t \cdot \hat{\cdot}_1 [; \lambda x^B.x, \lambda z^A.t \cdot \hat{\cdot}_2 [z; \lambda x^B.x]]$. Then $\pi_2 \{a, b; \}^\wedge \longrightarrow_a^+ b$.

609 An interesting feature is that the reduction rules for our non-optimized calculus are not
610 Church-Rosser, as we have already indicated in Example 30 and also in Example 23. On the
611 other hand, the optimized rules for standard intuitionistic proposition logic are know to be
612 Church-Rosser. We look into the case for \wedge in more detail.

► **Example 40.** The set of full rules for \wedge , see Example 30, is not Church-Rosser as the following concrete example shows. Suppose we have $\vdash p : D$ and $\vdash q : D$, where p and q are different.

$$\frac{\begin{array}{c} a : A \vdash a : A \quad b : B \vdash b : B \\ \hline a : A, b : B \vdash \{a, b; \}^\wedge : A \wedge B \end{array} \quad \begin{array}{c} x : A \vdash p : D \quad y : B \vdash q : D \\ \hline \{a, b; \}^\wedge \cdot \hat{\cdot}_1 [; \lambda x^A.p, \lambda y^B.q] \end{array}}$$

This term reduces to both p and q , which are distinct terms of type D . The crucial point is in the rule for $\cdot \hat{\cdot}_1 [; -]$ that admits a choice:

$$\frac{\vdash t : A \wedge B \quad x : A \vdash p : D \quad y : B \vdash q : D}{\vdash t \cdot \hat{\cdot}_1 [; \lambda x.p, \lambda y.q] : D}$$

613 For $t = \{a, b; \}^\wedge$ we can either select the “ A -case” or the “ B -case”.

614 We have shown how the optimized rules can be explained in terms of the full rules, but
615 we can also do the opposite: interpret the full rules for \wedge of Example 30 in terms of π_1 and
616 π_2 . Then we get

$$617 \quad t \cdot \hat{\cdot}_1 [; \lambda x^A.p, \lambda y^B.q] := p[x := \pi_1 t]$$

$$618 \quad t \cdot \hat{\cdot}_2 [a'; \lambda y^B.q] := q[y := \pi_2 t]$$

$$619 \quad t \cdot \hat{\cdot}_3 [b'; \lambda x^A.p] := p[x := \pi_1 t]$$

620 where in the first case we could also have chosen $q[y := \pi_2 t]$. We observe that the non-
621 determinism in the full rules is resolved by a choice we make in the translation of the first
622 \wedge -elimination.

623 ► **Example 41.** We now look into the optimized rules for implication of Definition 13. The
 624 full rules have been treated in Example 31. We want to define the following terms.

$$\frac{x : A \vdash p : A \rightarrow B}{\vdash \{ ; \lambda x^A . p \}_1^{\rightarrow\circ} : A \rightarrow B} \quad \frac{\vdash b : B}{\vdash \{ b ; \}_2^{\rightarrow\circ} : A \rightarrow B} \quad \frac{\vdash t : A \rightarrow B \quad \vdash a : A}{\vdash t \square^{\rightarrow} [a] : B}$$

625 These can be defined from the terms in Example 31 via the optimizations of Definition
 626 36 as follows.

$$\begin{aligned} 627 \quad \{ ; \lambda x^A . p \}_1^{\rightarrow\circ} &:= \{ ; \lambda x^A . p, \lambda z . \{ z ; \lambda x^A . p \}_2^{\rightarrow} \}_1^{\rightarrow} \\ 628 \quad \{ b ; \}_2^{\rightarrow\circ} &:= \{ b ; \lambda z^A . \{ z, b ; \}_3^{\rightarrow} \}_2^{\rightarrow} \\ 629 \quad t \square^{\rightarrow} [a] &:= t \cdot^{\rightarrow} [a ; \lambda z . z] \end{aligned}$$

630 These obey the following reductions.

$$\begin{aligned} 631 \quad \{ ; \lambda x^A . p \}_1^{\rightarrow\circ} \square^{\rightarrow} [a] &= \{ ; \lambda x^A . p, \lambda z . \{ z ; \lambda x^A . p \}_2^{\rightarrow} \}_1^{\rightarrow} \cdot^{\rightarrow} [a ; \lambda z . z] \\ 632 &\longrightarrow_a p[x := a] \cdot^{\rightarrow} [a ; \lambda z . z] \\ 633 &= p[x := a] \square^{\rightarrow} [a] \\ 634 \quad \{ b ; \}_2^{\rightarrow\circ} \square^{\rightarrow} [a] &:= \{ b ; \lambda z^A . \{ z, b ; \}_3^{\rightarrow} \}_2^{\rightarrow} \square^{\rightarrow} [a] \\ 635 &\longrightarrow_a b \\ 636 \quad \{ b ; \}_2^{\rightarrow\circ} \square^{\rightarrow} [a] &:= \{ b ; \lambda z^A . \{ z, b ; \}_3^{\rightarrow} \}_2^{\rightarrow} \square^{\rightarrow} [a] \\ 637 &\longrightarrow_a \{ a, b ; \}_3^{\rightarrow} \square^{\rightarrow} [a] \\ 638 &\longrightarrow_a b \end{aligned}$$

639 These are the exact reduction rules one would expect for these terms. We can again translate
 640 these to the well-known β -rules, that we will define in Definition 47.

641 The definition of the standard rule for \rightarrow -introduction essentially uses the \square construction,
 642 which has a somewhat special behaviour under normalization, as we have seen in Remark 5
 643 and Lemma 38. Let's look at an example to emphasize this.

644 ► **Example 42.** Consider the following proof.

$$\frac{\frac{t : A \rightarrow B \rightarrow C \quad \vdash a : A}{t \square^{\rightarrow} [a] : B \rightarrow C} \quad \vdash b : B}{t \square^{\rightarrow} [a] \square^{\rightarrow} [b] : C}$$

645 If t is not an introduction term ($t \neq \{ \lambda x . q \}^{\rightarrow}$), then this is not a redex with the optimized
 646 rules. However, in case \square is a defined term-construction, this term is reducible:

$$t \square^{\rightarrow} [a] \square^{\rightarrow} [b] \longrightarrow_b t \cdot^{\rightarrow} [a ; \lambda z^{B \rightarrow C} . z \square^{\rightarrow} [b]].$$

To clarify, the derivation for this term is:

$$\frac{\vdash t : A \rightarrow B \rightarrow C \quad \vdash a : A \quad \frac{z : B \rightarrow C \vdash z : B \rightarrow C \quad \vdash b : B}{z : B \rightarrow C \vdash z \square^{\rightarrow} [b] : C}}{t \cdot^{\rightarrow} [a ; \lambda z^{B \rightarrow C} . z \square^{\rightarrow} [b]] : C}$$

647 ► **Lemma 43.** The translation of an \longrightarrow_a step in the optimized calculus translates to a
 648 (possibly multistep) \longrightarrow_a step in the original calculus λ^C .

649 **Proof.** We show two cases:

650 1. If $\{\bar{t}; \overline{\lambda y.v}\}_{r_3, r_4}^\circ \odot_{r_1, r_2} [\bar{p}; \overline{\lambda x.q}] \rightarrow_a R$ (using the optimized rules) and
 651 $\{\bar{t}; \overline{\lambda y.r}\}_{r_3, r_4}^\circ \odot_{r_1, r_2} [\bar{p}; \overline{\lambda x.q}]$ translates to T in the original calculus λ^C , then there is a
 652 term T' such that $T \rightarrow_a^+ T'$ and R translates to T' in λ^C . Here \rightarrow_a^+ denotes a non-zero
 653 sequence of reductions.

654 In this case the translation T is as follows. $T = M \cdot \overline{[\bar{p}; \lambda x.q, \lambda z.M \cdot \overline{[\bar{p}, \bar{z}; \overline{\lambda x.q}]}]}$, where
 655 we abbreviate $M := \{\bar{t}; \overline{\lambda y.v}, \lambda z.\{\bar{t}, \bar{z}; \overline{\lambda y.v}\}\}$. There are two possible cases for the
 656 reduction.

657 = Case $\{\bar{t}; \overline{\lambda y.v}\}_{r_3, r_4}^\circ \odot_{r_1, r_2} [\bar{p}; \overline{\lambda x.q}] \rightarrow_a q_\ell[x_\ell := t_j]$. Then $T \rightarrow_a q_\ell[x_\ell := t_j]$ and
 658 we are done.

659 = Case $\{\bar{t}; \overline{\lambda y.v}\}_{r_3, r_4}^\circ \odot_{r_1, r_2} [\bar{p}; \overline{\lambda x.q}] \rightarrow_a v_i[y_i := p_k] \odot_{r_1, r_2} [\bar{p}; \overline{\lambda x.q}]$. Then

$$660 \quad T \rightarrow_a v_i[y_i := p_k] \cdot \overline{[\bar{p}; \lambda x.q, \lambda z.M \cdot \overline{[\bar{p}, \bar{z}; \overline{\lambda x.q}]}]}$$

$$661 \quad \rightarrow_a v_i[y_i := p_k] \cdot \overline{[\bar{p}; \lambda x.q, \lambda z.v_i[y_i := p_k] \cdot \overline{[\bar{p}, \bar{z}; \overline{\lambda x.q}]}]}$$

662 and we are done.

663 2. If $\{\bar{t}; \overline{\lambda y.v}\}_{r_2, r_3}^\circ \odot_{r_1} [\bar{p}] \rightarrow_a R$ and $\{\bar{t}; \overline{\lambda y.r}\}_{r_2, r_3}^\circ \odot_{r_1} [\bar{p}]$ translates to T in the original
 664 calculus λ^C , then there is a term T' such that $T \rightarrow_a^+ T'$ and R translates to T' in λ^C .

665 Now the translation T is as follows. $T = \{\bar{t}; \overline{\lambda y.v}, \lambda z.\{\bar{t}, \bar{z}; \overline{\lambda y.v}\}\} \cdot \overline{[\bar{p}; \lambda z.z]}$. There is
 666 one possibility for the reduction.

667 = Case $\{\bar{t}; \overline{\lambda y.v}\}_{r_2, r_3}^\circ \odot_{r_1} [\bar{p}] \rightarrow_a v_i[y_i := p_k] \odot_{r_1} [\bar{p}]$. Then

$$668 \quad T \rightarrow_a v_i[y_i := p_k] \cdot \overline{[\bar{p}; \lambda z.z]}$$

669 and we are done.

670

671 As mentioned, Schroeder-Heister[17] has proposed another elimination rule for \wedge which is
 672 slightly different from ours. Von Plato [23] calls this *general elimination* while Tennant [21]
 673 calls it *parallel elimination*. We call it parallel \wedge -elimination and give it in typed λ -calculus
 674 format.

► **Definition 44.** We define the *parallel \wedge -elimination rule* as follows

$$\frac{\Gamma \vdash t : A \wedge B \quad \Gamma, x : A, y : B \vdash q : D}{\Gamma \vdash t \cdot^{\text{par}} [\lambda x, y.q] : D} \wedge\text{-el}$$

The reduction (detour conversion) rule associated with this rule is as follows.

$$\{a, b; \} \cdot^{\text{par}} [\lambda x, y.q] \rightarrow_{\text{par}} q[x := a, y := b].$$

675 We show that this elimination rule can be translated in terms of ours and that reduction
 676 is preserved.

► **Definition 45.** We translate the parallel \wedge -elimination rule of Definition 44 by defining it
 in terms of the optimized terms for \wedge of Example 39. We consider the following optimized
 rules, the first of which was given explicitly in Example 39.

$$\frac{\Gamma \vdash t : A \wedge B \quad \Gamma, x : A \vdash q : D}{\Gamma \vdash t \odot_a^\wedge [; \lambda x.q] : D} \quad \frac{\Gamma \vdash t : A \wedge B \quad \Gamma, y : B \vdash q : D}{\Gamma \vdash t \odot_b^\wedge [; \lambda y.q] : D}$$

Now define

$$t \cdot^{\text{par}} [\lambda x, y.q] := t \odot_a^\wedge [; \lambda x. t \odot_b^\wedge [; \lambda y.q]].$$

677 ► **Lemma 46.** *The defined term $t \cdot^{\text{par}} [\lambda x, y.q]$ is of the right type and the translation of an*
 678 *\rightarrow_{par} step in the calculus with the parallel \wedge -elimination rule translates to multistep \rightarrow_a*
 679 *in the original calculus λ^C .*

Proof. Given $\Gamma \vdash t : A \wedge B$ and $\Gamma, x : A, y : B \vdash q : D$, we have

$$\frac{\Gamma \vdash t : A \wedge B \quad \frac{\Gamma, x : A \vdash t : A \wedge B \quad \Gamma, x : A, y : B \vdash q : D}{\Gamma, x : A \vdash t \odot_b^\wedge [; \lambda y.q] : D}}{\Gamma \vdash t \odot_a^\wedge [; \lambda x. t \odot_b^\wedge [; \lambda y.q]] : D}$$

680 The reduction can easily be verified:

$$\begin{aligned} 681 \quad \{a, b ; \}^\wedge .\text{par} [\lambda x, y.q] &:= \{a, b ; \}^\wedge \odot_a^\wedge [; \lambda x. \{a, b ; \}^\wedge \odot_b^\wedge [; \lambda y.q]] \\ 682 \quad &\longrightarrow_a \{a, b ; \}^\wedge \odot_b^\wedge [; \lambda y.q[x := a]] \\ 683 \quad &\longrightarrow_a q[x := a, y := b]. \end{aligned}$$

684

685 We define the standard rule for \rightarrow -introduction and show that this introduction rule can
686 be translated in terms of ours and that the reduction is preserved.

► **Definition 47.** We define the *standard rule for \rightarrow -introduction* as follows, where we describe it using terms.

$$\frac{\Gamma, x : A \vdash q : B}{\Gamma \vdash \{\lambda x.q\}^\rightarrow : A \rightarrow B} \rightarrow\text{-in}$$

The reduction rule associated with this term is as follows.

$$\{\lambda x.q\}^\rightarrow \square^\rightarrow [a] \longrightarrow_s q[x := a],$$

687 where $t \square^\rightarrow [a]$ is the optimized elimination rule from Example 41.

► **Definition 48.** We define the standard \rightarrow -introduction rule in terms of optimized \rightarrow -rules (Example 41) as follows. Given $\Gamma, x : A \vdash q : B$ we define

$$\{\lambda x.q\}^\rightarrow := \{ ; \lambda x. \{q ; \}_2^{\rightarrow\circ} \}_1^{\rightarrow\circ}.$$

688 ► **Lemma 49.** *The translation of $\{\lambda x.q\}^\rightarrow$ is well-typed and the translation of an \longrightarrow_s step*
689 *in the calculus with the standard rule for \rightarrow translates to multistep \longrightarrow_a in the original*
690 *calculus λ^c .*

691 **Proof.** The well-typedness is easily verified:

$$\frac{\frac{x : A \vdash q : B}{x : A \vdash \{q ; \}_2^{\rightarrow\circ} : A \rightarrow B}}{\vdash \{ ; \lambda x^A. \{q ; \}_2^{\rightarrow\circ} \}_1^{\rightarrow\circ} : A \rightarrow B}$$

For the reduction:

$$\{ ; \lambda x^A. \{q ; \}_2^{\rightarrow\circ} \}_1^{\rightarrow\circ} \cdot^\rightarrow [a ;] \longrightarrow_a \{q[x := a] ; \}_2^{\rightarrow\circ} \cdot^\rightarrow [a ;] \longrightarrow_a q[x := a].$$

692

693 We define the traditional rule for \neg -introduction and show that it can be translated in
694 terms of ours and that detour conversion is preserved.

3:30 Proof terms for generalized natural deduction

► **Definition 50.** We define the *traditional rules* for \neg , the introduction and the elimination rule, as follows, where we describe them using terms.

$$\frac{\Gamma, x : A \vdash t : \neg B \quad \Gamma, y : A \vdash q : B}{\Gamma \vdash \{\lambda x.t, \lambda y.q\}^t : \neg A} \quad \frac{\Gamma \vdash t : \neg A \quad \Gamma \vdash a : A}{\Gamma \vdash t \cdot^\neg [a ;] : D}$$

The reduction rule associated with these terms is as follows.

$$\{\lambda x^A.t, \lambda y^A.q\}^t \cdot^\neg [a ;] \longrightarrow_{\neg} t[x := a] \cdot^\neg [q[y := a] ;].$$

► **Example 51.** The rules for negation that we derive from our general Definition 27 are the following.

$$\frac{\Gamma, x : A \vdash q : \neg A}{\Gamma \vdash \{ ; \lambda x.q\}^\neg : \neg A} \quad \frac{\Gamma \vdash t : \neg A \quad \Gamma \vdash a : A}{\Gamma \vdash t \cdot^\neg [a ;] : D}$$

695 with reduction

$$\{ ; \lambda x^A.q\}^\neg \cdot^\neg [a ;] \longrightarrow_a q[x := a] \cdot^\neg [a ;].$$

696 We see that the elimination rule for \neg in Example 51 is the same as the traditional one.
697 The traditional introduction rule for \neg is definable.

► **Definition 52.** We define the traditional \neg -introduction rule in terms of the one of Example 51 as follows. Given $\Gamma, x : A \vdash t : \neg B$ and $\Gamma, y : A \vdash q : B$ we define

$$\{\lambda x^A.t, \lambda y^A.q\}^t := \{ ; \lambda x^A.t \cdot^\neg [q[y := x] ;]\}^\neg$$

698 ► **Lemma 53.** *The definition of $\{\lambda x.t, \lambda y.q\}^t$ is well-typed and a \longrightarrow_{\neg} step in the calculus*
699 *with the traditional rule for \neg translates to multistep \longrightarrow_a in the original calculus λ^C .*

Proof. For the well-typedness:

$$\frac{\Gamma, x : A \vdash t : \neg B \quad \frac{\Gamma, y : A \vdash q : B}{\Gamma, x : A \vdash q[y := x] : B} \neg\text{-el}}{\Gamma, x : A \vdash t \cdot^\neg [q[y := x] ;] : \neg A} \neg\text{-in}$$

$$\frac{\Gamma, x : A \vdash t \cdot^\neg [q[y := x] ;] : \neg A}{\Gamma \vdash \{ ; \lambda x^A.t \cdot^\neg [q[y := x] ;]\}^\neg : \neg A} \neg\text{-in}$$

For the reduction:

$$\{ ; \lambda x^A.t \cdot^\neg [q[y := x] ;]\}^\neg \cdot^\neg [a ;] \longrightarrow_a t[x := a] \cdot^\neg [q[x := a] ;].$$

700

701 As a final example, we give the proof-terms for the optimized rules of nand-logic, as
702 described in Definition 14.

► **Example 54.** The proof-terms for nand-logic are

$$\frac{x : A \vdash p : A \uparrow B}{\vdash \{ ; \lambda x^A.p\}^\uparrow : A \uparrow B} \quad \frac{y : B \vdash q : A \uparrow B}{\vdash \{ ; \lambda y^B.q\}^\uparrow : A \uparrow B} \quad \frac{\vdash t : A \uparrow B \quad \vdash a : A \quad \vdash b : B}{\vdash t \cdot^\uparrow [a, b ;] : D}$$

703 with reduction rules

704 $\{ ; \lambda x^A.p\}^\uparrow \cdot^\uparrow [a, b ;] \longrightarrow_a p[x := a] \cdot^\uparrow [a, b ;]$

705 $\{ ; \lambda y^B.q\}^\uparrow \cdot^\uparrow [a, b ;] \longrightarrow_a q[y := b] \cdot^\uparrow [a, b ;]$

706 This time we have a situation where a permutation conversion actually reduces the size of a
707 term considerably. Suppose $t : A \uparrow B$ and $a : A, b : B, c : C, d : D$. Then we have

$$\frac{\frac{\vdash t : A \uparrow B \quad \vdash a : A \quad \vdash b : B}{\vdash t \cdot \uparrow [a, b ;] : C \uparrow D} \quad \vdash c : C \quad \vdash d : D}{t \cdot \uparrow [a, b ;] \cdot \uparrow [c, d ;] : E}$$

We have

$$t \cdot \uparrow [a, b ;] \cdot \uparrow [c, d ;] \longrightarrow_b t \cdot \uparrow [a, b ;]$$

708 which is of type E , and we see that the superfluous second **nand**-elimination rule has been
709 removed.

As another example, we can give a proof-term of $A \vee \neg A^\uparrow$, the proposition in **nand**-logic that we have shown to be provable after the proof of Proposition 17. It's proof-term is

$$\{ ; \lambda x. \{ ; \lambda y. y \cdot \uparrow [x, x ;] \}^\uparrow \}^\uparrow : (A \uparrow A) \uparrow (\neg A \uparrow \neg A)$$

710 6 Normalization

711 In this section we prove that \longrightarrow_a and \longrightarrow_b are both strongly normalizing (SN). We also
712 give a proof of weak normalization (WN) of the combination of \longrightarrow_a and \longrightarrow_b . As usual,
713 SN states that there are no terms that have an infinite reduction path, and WN states that
714 for each term there is a reduction path that leads to a normal form. For the proof of WN we
715 describe an actual procedure for finding a normal form of a term.

716 ► **Theorem 55.** *The reduction \longrightarrow_b is strongly normalizing.*

717 **Proof.** We define a measure $| - |$ from terms to natural numbers that decreases with every
718 reduction step. For notational convenience we suppress the reference to the derivation rule r .

$$\begin{aligned} 719 \quad |x| &:= 1 \\ 720 \quad |\{\bar{p} ; \overline{\lambda y. q}\}| &:= \Sigma |p_i| + \Sigma |q_j| \\ 721 \quad |t \cdot [\bar{s} ; \overline{\lambda y. u}]| &:= |t|(2 + \Sigma |s_k| + \Sigma |u_\ell|) \end{aligned}$$

722 It can easy be verified that, if $t_0 \longrightarrow_b t_1$, then $|t_0| > |t_1|$, so \longrightarrow_b is strongly normalizing. ◀

723 ► **Corollary 56.** *The reduction \longrightarrow_b for the optimized rules of Definition 36, the standard
724 rule for \rightarrow -elimination of Definition 47, the parallel \wedge -elimination rule of Definition 44 and
725 the traditional rule for \neg -elimination of Definition 50 are strongly normalizing.*

726 **Proof.** The same metrics as in the proof of Theorem 55 applies. For the parallel reduction,
727 define $|t \cdot \text{par} [\lambda x, y. q]| := |t|(2 + |q|)$.
728 ◀

729 6.1 Strong Normalization of the detour conversion

730 We now prove strong normalization for \longrightarrow_a by adapting the well-known *saturated sets*
731 *method* of Tait [20] and Girard [8] to our calculus. Recall that **Term** is the set of all untyped
732 proof-terms. (Definition 27.) We write **SN** for the set of strongly normalizing (untyped)
733 terms and we write **Var** for the set of variables.

734 ► **Definition 57.** 1. The set **Neut** of *neutral terms* is defined by

735 a. $\text{Var} \subseteq \text{Neut}$,

- 736 b. $t \cdot [\bar{p}; \overline{\lambda y.q}] \in \text{Neut}$ for all $t \in \text{Neut}$ and $\bar{p}, \overline{\lambda y.q} \in \text{SN}$.
- 737 2. The term t does a *key reduction* to t' , notation $t \rightarrow_a^k t'$, in case
- 738 a. t is a redex itself (according to Definition 29) and t' is its reduct,
- 739 b. $t = t_0 \cdot [\bar{p}; \overline{\lambda y.q}]$, $t' = t_1 \cdot [\bar{p}; \overline{\lambda y.q}]$ and $t_0 \rightarrow_a^k t_1$.
- 740 3. A set $X \subseteq \text{Term}$ is *saturated* ($X \in \text{SAT}$) if it satisfies the following properties
- 741 a. $X \subseteq \text{SN}$,
- 742 b. $\text{Neut} \subseteq X$
- 743 c. X is closed under *key-redex expansion*: if $t \in \text{SN}$ and $\forall q(t \rightarrow_a^k q \Rightarrow q \in X)$, then
- 744 $t \in X$.
4. For a connective c of arity n and $X_1, \dots, X_n \in \text{SAT}$ we define the set $c(X_1, \dots, X_n)$ as follows. Assume that r_1, \dots, r_m are the elimination rules for c .

$$c(X_1, \dots, X_n) := \{t \mid \forall r_i \in \{r_1, \dots, r_m\} \\ \forall D \in \text{SAT}, \forall \bar{p}, \bar{q} \in \text{Term} \\ \forall k(p_k \in X_k) \wedge (\forall \ell \forall u_\ell \in X_\ell (q_\ell[y_\ell := u_\ell] \in D)) \implies t \cdot_{r_i} [\bar{p}; \overline{\lambda y.q}] \in D\}$$

745 In the definition of $c(X_1, \dots, X_n)$ it should be clear that we quantify over all elimination

746 rules for the connective c . In the quantification $\forall \bar{p}, \bar{q} \in \text{Term}$ we could also quantify over

747 $\forall \bar{p}, \bar{q} \in \text{SN}$: it amounts to the same because the additional conditions $\forall k(p_k \in X_k)$ and

748 $\forall \ell \forall u_\ell \in X_\ell (q_\ell[y_\ell := u_\ell] \in D)$ imply that $\bar{p}, \bar{q} \in \text{SN}$.

749 ► **Lemma 58.** *If $X_1, \dots, X_n \in \text{SAT}$, then $c(X_1, \dots, X_n) \in \text{SAT}$.*

750 **Proof.** We check the 3 conditions for $c(X_1, \dots, X_n)$. Suppose $X_1, \dots, X_n \in \text{SAT}$.

- 751 a. That $c(X_1, \dots, X_n) \subseteq \text{SN}$ follows directly from the fact that if $t \in c(X_1, \dots, X_n)$, then
- 752 $t \cdot [\bar{p}; \overline{\lambda x.q}] \in D$ and $D \subseteq \text{SN}$, so $t \cdot [\bar{p}; \overline{\lambda x.q}] \in \text{SN}$, so $t \in \text{SN}$.
- 753 b. For $t \in \text{Neut}$ and $D \in \text{SAT}$ and $\bar{p}, \bar{q} \in \text{SN}$ with $\forall k(p_k \in X_k)$ and $\forall \ell \forall u_\ell \in X_\ell (q_\ell[y_\ell :=$
- 754 $u_\ell] \in D)$, we have $t \cdot_{r_i} [\bar{p}; \overline{\lambda y.q}] \in \text{Neut} \subseteq D$, so we can conclude that $t \in c(X_1, \dots, X_n)$.
- 755 c. Suppose $t \in \text{SN}$ and $\forall t'(t \rightarrow_a^k t' \Rightarrow t' \in c(X_1, \dots, X_n))$ (*). Let r_i be a rule for c and
- 756 let $D \in \text{SAT}$, $\bar{p}, \bar{q} \in \text{Term}$ with $\forall k(p_k \in X_k)$ and $\forall \ell \forall u_\ell \in X_\ell (q_\ell[y_\ell := u_\ell] \in D)$. For all t'
- 757 with $t \rightarrow_a^k t'$ we have $t \cdot_{r_i} [\bar{p}; \overline{\lambda y.q}] \rightarrow_a^k t' \cdot_{r_i} [\bar{p}; \overline{\lambda y.q}]$ and $t' \cdot_{r_i} [\bar{p}; \overline{\lambda y.q}] \in D$ by (*).
- 758 So, $t \cdot_{r_i} [\bar{p}; \overline{\lambda y.q}] \in D$ and so $t \in c(X_1, \dots, X_n)$.
- 759 ◀

760 We use the saturated sets as a semantics for types: if A is a type, $\langle A \rangle$ will be a saturated

761 set. The simplest way to do this is to interpret all type variables (proposition letters) as the

762 set SN , which is indeed a saturated set.

763 ► **Definition 59.** For A a type, we define $\langle A \rangle$ by induction on A as follows.

- 764 ■ $\langle A \rangle := \text{SN}$ if A is a proposition letter.
- 765 ■ $c(A_1, \dots, A_n) := c(\langle A_1 \rangle, \dots, \langle A_n \rangle)$, where the right hand side is the interpretation of the
- 766 connective c on saturated sets, as given in Definition 57, case (4).

767 We will often confuse A and $\langle A \rangle$, to avoid notational overhead, and just identify the

768 proposition A with its interpretation as a saturated set $\langle A \rangle$.

769 ► **Definition 60.** Given a context Γ , a map (valuation) $\rho : \text{Var} \rightarrow \text{Term}$ satisfies Γ , notation

770 $\rho \models \Gamma$, in case $\rho(x) \in \langle A \rangle$ for all $x : A \in \Gamma$.

771 If $t \in \text{Term}$ and $\rho : \text{Var} \rightarrow \text{Term}$, we write $\langle t \rangle_\rho$ for t where ρ has been carried out as a

772 substitution on t .

773 A valuation $\rho : \text{Var} \rightarrow \text{Term}$ is only relevant for a finite number of variables: those that
 774 are declared in the context Γ under consideration. So we will always assume that $\rho(x) \neq x$
 775 only for a finite number of $x \in \text{Var}$. Those x we call the *support* of ρ . When applying ρ as a
 776 substitution to a term t we may need to “go under a λ ”, e.g. when applying ρ to $\{\bar{p}; \overline{\lambda x. q}\}$
 777 In this case we always assume that the bound variable is not in the support of ρ . (We can
 778 always rename it.)

779 ► **Lemma 61.** *If $\Gamma \vdash t : A$, and $\rho \models \Gamma$, then $\langle t \rangle_\rho \in \langle A \rangle$.*

780 **Proof.** By induction on the derivation of $\Gamma \vdash t : A$. Suppose $\rho \models \Gamma$. For the (axiom) case, it
 781 is trivial. We ignore ρ for the rest of the proof, as it gives a lot of notational overhead, so we
 782 just write t for $\langle t \rangle_\rho$.

■ Suppose $\Phi = c(A_1, \dots, A_n)$ and

$$\frac{\dots \Gamma \vdash s_j : A_j \dots \dots \Gamma, x_i : A_i \vdash t_i : \Phi \dots}{\Gamma \vdash \{\bar{s}; \overline{\lambda x. t}\}_r : \Phi} \text{in}$$

783 Let r' be a rule for c , $D \in \text{SAT}$, $\bar{p}, \bar{q} \in \text{Term}$ with $\forall k (p_k \in A_k)$ and $\forall \ell \forall u_\ell \in A_\ell (q_\ell[y_\ell :=$
 784 $u_\ell] \in D)$. For $\{\bar{s}; \overline{\lambda x. t}\}_r \cdot_{r'} [\bar{p}; \overline{\lambda y. q}]$ there are the following possible key-reductions:

$$785 \quad \{\bar{s}; \overline{\lambda x. t}\}_r \cdot_{r'} [\bar{p}; \overline{\lambda y. q}] \longrightarrow_a^k q_l[y_l := s_j] \quad (1)$$

$$786 \quad \{\bar{s}; \overline{\lambda x. t}\}_r \cdot_{r'} [\bar{p}; \overline{\lambda y. q}] \longrightarrow_a^k t_i[x_i := p_k] \cdot_{r'} [\bar{p}; \overline{\lambda y. q}] \quad (2)$$

787 In case (1), $q_l[y_l := s_j] \in D$ by the assumption and the induction hypothesis. In case (2),
 788 $t_i[x_i := p_k] \in \Phi$ by the induction hypothesis and so $t_i[x_i := p_k] \cdot_{r'} [\bar{p}; \overline{\lambda y. q}] \in D$ by the
 789 definition of $\Phi = c(A_1, \dots, A_n)$ as a saturated set. So, $\{\bar{s}; \overline{\lambda x. t}\}_r \cdot_{r'} [\bar{p}; \overline{\lambda y. q}] \in \text{SN}$ and
 790 all its key reductions are in D , so the term is in D . Therefore, $\{\bar{s}; \overline{\lambda x. t}\}_r \in \Phi$.

■ Suppose $\Phi = c(A_1, \dots, A_n)$ and

$$\frac{\Gamma \vdash t : \Phi \quad \dots \Gamma \vdash p_k : A_k \dots \quad \dots \Gamma, y_\ell : A_\ell \vdash q_\ell : D}{\Gamma \vdash t \cdot_r [\bar{p}; \overline{\lambda y. q}] : D} \text{el}$$

791 Then $t \cdot_r [\bar{p}; \overline{\lambda y. q}] = t \cdot_r [\bar{p}; \overline{\lambda y. q}] \in D$ by $t \in \Phi = c(A_1, \dots, A_n)$ and the definition of
 792 $c(A_1, \dots, A_n)$ as a saturated set and the induction hypothesis.

793 ◀

794 The following is now an immediate corollary by taking $\rho(x) := x$ for all $x \in \text{Var}$. Because
 795 $\text{Var} \subseteq \text{Neut} \subseteq \langle A \rangle$, we know that $\rho \models \Gamma$. So, if $\Gamma \vdash t : A$, then $\langle t \rangle_\rho = t \in \langle A \rangle \subseteq \text{SN}$.

796 ► **Theorem 62.** *The reduction \longrightarrow_a is strongly normalizing: all \longrightarrow_a -reductions on proof*
 797 *terms are finite.*

798 ► **Corollary 63.** *The reduction \longrightarrow_a for the optimized rules of Definition 36, the parallel*
 799 *\wedge -elimination rule of Definition 44, the standard \rightarrow -introduction of Definition 47 and the*
 800 *traditional rule for \neg -elimination of Definition 50 are strongly normalizing.*

801 **Proof.** By Theorem 62 and the fact that reduction is preserved by the translation: Lemmas
 802 43, 46 and 49. ◀

803 6.2 Weak Normalization of conversion

804 We now give a strategy for finding a normal form for the combined \longrightarrow_{ab} reduction, the union
 805 of \longrightarrow_a and \longrightarrow_b . This proves that \longrightarrow_{ab} is weakly normalizing and it also gives a concrete
 806 procedure for finding a normal form. Due to the fact that, in general, reduction is not

807 confluent, this normal form is not unique, but it does yield *decidability* via the *sub-formula*
 808 *property*. The weak normalization proof follows the well-known idea, originally due to Turing
 809 (see [5]) for simple type theory, to *contract the innermost redex of highest rank*.

810 ► **Definition 64.** We define the *rank of a formula* A , $\text{rk}(A)$ as follows.

811 ■ $\text{rk}(A) := 1$ if A is a proposition letter.

812 ■ $\text{rk}(c(A_1, \dots, A_n)) := 1 + \max\{\text{rk}(A_1), \dots, \text{rk}(A_n)\}$ if c is a connective of arity n .

813 We define the *rank of a redex* as follows.

814 ■ The rank of $\{\overline{p}; \overline{\lambda x.q}\}_{r'} \cdot_r [\overline{s}; \overline{\lambda y.r}]$ is the rank of the type of $\{\overline{p}; \overline{\lambda x.q}\}_{r'}$.

815 ■ The rank of $(t \cdot_{r'} [\overline{p}; \overline{\lambda x.q}]) \cdot_r [\overline{s}; \overline{\lambda y.r}]$ is the rank of the type of $t \cdot_{r'} [\overline{p}; \overline{\lambda x.q}]$.

816 We will sometimes mark the redex with its type Φ such that $\text{rk}(\Phi)$ is the rank of the
 817 redex. We do this by writing Φ as a superscript to the elimination constructor. To clarify,
 818 we summarize again the possible reduction steps of the form \longrightarrow_a and \longrightarrow_b .

819 ► **Notation 65.** From Definition 29, we have the reduction \longrightarrow_a and from Definition 32 we
 820 have the reduction \longrightarrow_b . We introduce the following notation.

$$\begin{aligned}
 821 \quad & \{\overline{p}, \overline{p_j}; \overline{\lambda x.q}\} \cdot^\Phi [\overline{s}; \overline{\lambda y.r}, \overline{\lambda y_\ell.r_\ell}] \longrightarrow_{a1} r_\ell[y_\ell := p_j] \\
 822 \quad & \{\overline{p}; \overline{\lambda x.q}, \overline{\lambda x_i.q_i}\} \cdot^\Phi [\overline{s}, \overline{s_k}; \overline{\lambda y.r}] \longrightarrow_{a2} q_i[x_i := s_k] \cdot^\Phi [\overline{s}, \overline{s_k}; \overline{\lambda y.r}] \\
 823 \quad & (t \cdot [\overline{p}; \overline{\lambda x.q}]) \cdot^\Phi [\overline{s}; \overline{\lambda y.r}] \longrightarrow_b t \cdot [\overline{p}; \overline{\lambda x.(q \cdot^\Phi [\overline{s}; \overline{\lambda y.r}])]
 \end{aligned}$$

824 Here, the proviso's of Definition 29 apply, so the first is the “ $\ell = j$ case” which we will call
 825 \longrightarrow_{a1} , and the second is the “ $k = i$ case” which we will call \longrightarrow_{a2} .

826 We give two Lemmas that show that the creation of new redexes is limited.

827 ► **Lemma 66. 1.** If $t \longrightarrow_b t'$ by contracting a redex of $\text{rk}(\Phi)$ then the newly created redexes
 828 are also of $\text{rk}(\Phi)$.

829 2. Suppose $\{\overline{p}; \overline{\lambda x.q}, \overline{\lambda x_i.q_i}\} \cdot^\Phi [\overline{s}, \overline{s_k}; \overline{\lambda y.r}] \longrightarrow_{a2} q_i[x_i := s_k] \cdot^\Phi [\overline{s}, \overline{s_k}; \overline{\lambda y.r}]$. If $q_i[x_i := s_k]$
 830 is an introduction term (that is: $q_i[x_i := s_k]$ is of the form $\{\dots; \dots\}$), then q_i is an
 831 introduction term. Similarly, if $q_i[x_i := s_k]$ is an elimination term (that is: $q_i[x_i := s_k]$ is
 832 of the form $\dots \cdot [\dots; \dots]$), then q_i is an elimination term.

833 **Proof. 1.** If $t \longrightarrow_b t'$ by contracting a redex of $\text{rk}(\Phi)$, then t contains a sub-term

834 $s \cdot [\overline{p}; \overline{\lambda x.q}] \cdot^\Phi [\overline{u}; \overline{\lambda y.r}]$ which is contracted to $s \cdot [\overline{p}; \overline{\lambda x.q \cdot^\Phi [\overline{u}; \overline{\lambda y.r}]]$. The newly created
 835 redexes (if any) are all of $\text{rk}(\Phi)$.

836 2. Suppose $\{\overline{p}; \overline{\lambda x.q}, \overline{\lambda x_i.q_i}\} \cdot^\Phi [\overline{s}, \overline{s_k}; \overline{\lambda y.r}] \longrightarrow_{a2} q_i[x_i := s_k] \cdot^\Phi [\overline{s}, \overline{s_k}; \overline{\lambda y.r}]$. Then $q_i : \Phi$
 837 and $s_k : A_k$ which is a sub-formula of Φ , as $\Phi = c(A_1, \dots, A_n)$. If $q_i[x_i := s_k]$ is an
 838 introduction term, then either q_i is an introduction term itself or $q_i = x_i$ and s_k is an
 839 introduction term. The latter case can only occur if $s_k : \Phi$, but it is not, because its type
 840 is a sub-formula of Φ . So q_i is an introduction term. The case for $q_i[x_i := s_k]$ being an
 841 elimination term is similar. ◀

843 The Lemma states that both the newly created redexes due to \longrightarrow_b and \longrightarrow_{a2} are already
 844 “hidden” inside the term. We give a list of facts about redex creation and the ranks of redexes.

845 ► **Fact 67. 1.** A reduction step can produce more redexes either by (i) *copying existing*
 846 *redexes* or by (ii) *creating new redexes*. Copying occurs through substitution, in a reduction
 847 step \longrightarrow_{a1} or \longrightarrow_{a2} .

848 2. Creating new redexes happens in either one of the following ways.

- 849 a. When doing an \rightarrow_a step: in a sub-term $x \cdot [\bar{p}; \overline{\lambda y.q}]$, we substitute $\{\bar{s}; \overline{\lambda z.r}\}$ for x .
 850 This creates an a -redex of lower rank.
- 851 b. When doing an \rightarrow_a step: in a sub-term $x \cdot [\bar{p}; \overline{\lambda y.q}]$, we substitute $t \cdot [\bar{s}; \overline{\lambda z.r}]$ for x .
 852 This creates a b -redex of lower rank.
- 853 c. When $\{\bar{p}, \bar{p}_j; \overline{\lambda x.q}\} \cdot^\Phi [\bar{s}; \overline{\lambda y.r, \lambda y_\ell.r_\ell}] \rightarrow_{a1} r_\ell[y_\ell := p_j]$ where this term occurs as a
 854 sub-term: $r_\ell[y_\ell := p_j] \cdot^\Psi [\dots; \dots]$ and $r_\ell[y_\ell := p_j] = \{\dots; \dots\}$.
 855 This creates a new a -redex of unrelated rank.
- 856 d. When $\{\bar{p}, \bar{p}_j; \overline{\lambda x.q}\} \cdot^\Phi [\bar{s}; \overline{\lambda y.r, \lambda y_\ell.r_\ell}] \rightarrow_{a1} r_\ell[y_\ell := p_j]$ where this term occurs as a
 857 sub-term: $r_\ell[y_\ell := p_j] \cdot^\Psi [\dots; \dots]$ and $r_\ell[y_\ell := p_j] = \dots \cdot [\dots; \dots]$.
 858 This creates a new b -redex of unrelated rank.
- 859 e. When $\{\bar{p}; \overline{\lambda x.q, \lambda x_i.q_i}\} \cdot^\Phi [\bar{s}, \bar{s}_k; \overline{\lambda y.r}] \rightarrow_{a2} q_i[x_i := s_k] \cdot^\Phi [\bar{s}, \bar{s}_k; \overline{\lambda y.r}]$, where
 860 $q_i = \{\dots; \dots\}$.
 861 This creates a new a -redex of the same rank.
- 862 f. When $\{\bar{p}; \overline{\lambda x.q, \lambda x_i.q_i}\} \cdot^\Phi [\bar{s}, \bar{s}_k; \overline{\lambda y.r}] \rightarrow_{a2} q_i[x_i := s_k] \cdot^\Phi [\bar{s}, \bar{s}_k; \overline{\lambda y.r}]$, where
 863 $q_i = \dots \cdot [\dots; \dots]$.
 864 This creates a new b -redex of the same rank.
- 865 g. If $(t \cdot [\bar{p}; \overline{\lambda x.q}]) \cdot^\Phi [\bar{s}; \overline{\lambda y.r}] \rightarrow_b t \cdot [\bar{p}; \overline{\lambda x.(q \cdot^\Phi [\bar{s}; \overline{\lambda y.r}])}]$, where $q_i = \{\dots; \dots\}$.
 866 This creates a new a -redex (possibly more) of the same rank.
- 867 h. If $(t \cdot [\bar{p}; \overline{\lambda x.q}]) \cdot^\Phi [\bar{s}; \overline{\lambda y.r}] \rightarrow_b t \cdot [\bar{p}; \overline{\lambda x.(q \cdot^\Phi [\bar{s}; \overline{\lambda y.r}])}]$, where $q_i = \dots \cdot [\dots; \dots]$.
 868 This creates a new b -redex (possibly more) of the same rank.

869 Note that in the cases **e** and **f** of Fact 67 we use the second part of Lemma 66.

870 The idea is to contract an innermost redex of highest rank of a term in b -normal form
 871 (that is: a term that cannot do a \rightarrow_b -step). The advantage of b -normal forms is that cases
 872 **c** and **d** of the Fact 67 do not occur. (Because in these cases, the term one starts with is not
 873 in b -normal form.)

874 ► **Lemma 68.** *If f is a well-typed term in b -normal form that has one redex of maximum*
 875 *rank, say R , then f can be reduced to a term f' in b -normal form that has maximum rank*
 876 *below R .*

877 **Proof.** By induction on the size of f .

- 878 1. If $f = \{\bar{p}; \overline{\lambda x.q}\}$ or $f = x \cdot [\bar{p}; \overline{\lambda x.q}]$ or $f = \{\bar{p}; \overline{\lambda x.q}\} \cdot [\bar{s}; \overline{\lambda y.r}]$ and the redex of highest
 879 rank is inside \bar{p} , \bar{q} , \bar{s} or \bar{r} , then we are done by the induction hypothesis.
- 880 2. Suppose $f = \{\bar{p}; \overline{\lambda x.q}\} \cdot^\Phi [\bar{s}; \overline{\lambda y.r}]$ is itself a redex of highest rank, $\text{rk}(\Phi)$. We look at
 881 the possible ways in which a new redex may arise, following Fact 67. The cases **c**, **d**, **g**
 882 and **h** don't apply.

- 883 – For case **a**: the newly created redexes are of lower rank and the resulting term is in
 884 b -nf.
- 885 – For case **b**: the newly created redexes are of lower rank. The resulting term may not
 886 be in b -nf, but we can contract all the newly created b -redexes to obtain a b -normal
 887 form. According to Lemma 66, case (1), this does not create new redexes of higher
 888 rank, so we are done.
- 889 – For case **e**: $f = \{\bar{p}; \overline{\lambda x.q, \lambda x_i.q_i}\} \cdot^\Phi [\bar{s}, \bar{s}_k; \overline{\lambda y.r}] \rightarrow_{a2} q_i[x_i := s_k] \cdot^\Phi [\bar{s}, \bar{s}_k; \overline{\lambda y.r}]$ with
 890 $q_i = \{\dots; \dots\}$. By induction hypothesis, $q_i \cdot^\Phi [\bar{s}, \bar{s}_k; \overline{\lambda y.r}] \rightarrow g$ for some g in b -normal
 891 form with all redexes of lower rank. (Note that $q_i \cdot^\Phi [\bar{s}, \bar{s}_k; \overline{\lambda y.r}]$ is in b -normal form.)
 892 Then $q_i[x_i := s_k] \cdot^\Phi [\bar{s}, \bar{s}_k; \overline{\lambda y.r}] \rightarrow g[x_i := s_k]$ and due to the fact that the type of s_k
 893 is a sub-formula of Φ , this only contains new redexes of lower rank, so we are done.

894 ■ For case **f**: $f = \{\overline{p}; \overline{\lambda x.q}, \overline{\lambda x_i.q_i}\} \cdot^\Phi [\overline{s}, \overline{s_k}; \overline{\lambda y.r}] \rightarrow_{a2} q_i[x_i := s_k] \cdot^\Phi [\overline{s}, \overline{s_k}; \overline{\lambda y.r}]$
 895 with $q_i = t \cdot [\overline{u}; \overline{\lambda z.v}]$. If we take g to be the b -normal form of $q_i \cdot^\Phi [\overline{s}, \overline{s_k}; \overline{\lambda y.r}]$,
 896 this term contains disjoint sub-terms of the shape $\lambda w.d \cdot^\Phi [\overline{s}, \overline{s_k}; \overline{\lambda y.r}]$ that all have
 897 one maximal redex of rank R and that have length smaller than the length of f . By
 898 induction hypothesis, these can all be reduced to terms with only redexes of lower
 899 rank. Having done this, we obtain g as a reduct of $q_i \cdot^\Phi [\overline{s}, \overline{s_k}; \overline{\lambda y.r}]$ that is in b -normal
 900 form and contains only redexes of rank lower than R . To obtain f' , we notice that
 901 $f \rightarrow_{a2} q_i[x_i := s_k] \cdot^\Phi [\overline{s}, \overline{s_k}; \overline{\lambda y.r}] \rightarrow g'[x_i := s_k]$, which only contains b -redexes of
 902 lower rank, so we can take f' to be the b -normal form of $g'[x_i := s_k]$.

903 ◀

904 ► **Theorem 69.** *For any set of connectives \mathcal{C} , the reduction \rightarrow_{ab} of the calculus $\lambda^{\mathcal{C}}$ is weakly*
 905 *normalizing and we have a procedure to compute a normal form for a well-typed term.*

906 **Proof.** We consider the following measure $\mathbf{m}(-)$ terms: $\mathbf{m}(t) := (R, m)$, where R is the
 907 maximal rank of a redex in t and m is the number of redexes of rank R in t . We consider
 908 this measure under the lexicographic ordering.

909 Given a term t , we first compute its b -normal form, t_1 and consider $\mathbf{m}(t_1) = (R, m)$. Then
 910 we pick p , an innermost redex of maximal rank inside t_1 . Following Lemma 68, we reduce p
 911 to p' , in which all redexes are of rank below R . We do this reduction on t_1 , obtaining t_2 . (So
 912 $t_1 \rightarrow t_2$.) Notice that $\mathbf{m}(t_1) > \mathbf{m}(t_2)$. We continue in this way, obtaining a normal form of t ,
 913 because the lexicographic ordering is well-founded. ◀

914 We recall Lemma 35 which describes NF inductively, the set of terms in normal form. If t
 915 is in normal form, then t is of either one of the following three forms

- 916 1. t is a variable,
- 917 2. $t = \{\overline{p}; \overline{\lambda y.q}\}$, with all p_i and q_j in normal form,
- 918 3. $t = x \cdot [\overline{p}; \overline{\lambda y.q}]$, with x a variable and all p_i and q_j in normal form.

919 6.3 Corollaries of normalization

920 ► **Theorem 70.** *For any set of connectives \mathcal{C} , the calculus $\lambda^{\mathcal{C}}$ is consistent, that is: there*
 921 *are types A for which there is no closed term t with $\vdash t : A$.*

922 **Proof.** Take A to be a propositional variable and suppose $\vdash t : A$ with t in normal form.
 923 The three possible cases for t are given in Lemma 35, which we have recalled above. The
 924 first and third case are impossible, because t cannot contain any free variable. The second
 925 case is impossible, because an introduction term is always of a composite type. ◀

926 The calculus (and logic) $\lambda^{\mathcal{C}}$ also satisfies the sub-formula property.

927 ► **Theorem 71.** *Given a set of connectives \mathcal{C} , the calculus $\lambda^{\mathcal{C}}$ satisfies the sub-formula*
 928 *property, that is: if $\Gamma \vdash t : A$, then there is a term t' such that $\Gamma \vdash t' : A$ and all types of all*
 929 *sub-terms of t' are either sub-types of A or of some A_i for a declaration $x_i : A_i$ in Γ .*

930 **Proof.** If $\Gamma \vdash t : A$, then (by Theorem 69) there is a term t' in normal form with $\Gamma \vdash t' : A$.
 931 We use Lemma 35 and prove by induction on t' that “all types of all sub-terms of t' are either
 932 sub-types of A or of some A_i for a declaration $x_i : A_i$ in Γ ”. For simplicity we abbreviate
 933 this property to “ t' satisfies the sub-type property for $\Gamma; A$ ”.

934 ■ $t' = x$, a variable. Then we are done.

- 935 ■ $t' = \{\bar{p}; \overline{\lambda x.q}\}$, an introduction term. Then by induction hypothesis, all sub-terms of
 936 \bar{p} satisfy the sub-type property for $\Gamma; A_i$ for some A_i which is a sub-type of A . For
 937 the $\lambda x_j.q_j$ in $\overline{\lambda x.q}$, we have $\Gamma, x_j : A_j \vdash q_j : A$ for some A_j which is a sub-type of A .
 938 By induction hypothesis, for all j , all sub-terms of q_j satisfy the sub-type property for
 939 $\Gamma, x_j : A_j; A$. So all sub-terms of $\overline{\lambda x.q}$ satisfy the sub-type property for $\Gamma; A$ and we are
 940 done.
- 941 ■ $t' = x \cdot [\bar{p}; \overline{\lambda x.q}]$, an elimination term. Suppose $x : C$. Each p_i is of type B_i for some
 942 sub-type B_i of C , so the induction hypothesis yields that all sub-terms of \bar{p} satisfy the
 943 sub-type property for $\Gamma; A$. For the $\lambda x_j.q_j$ in $\overline{\lambda x.q}$, we have $\Gamma, x_j : B_j \vdash q_j : A$ for some
 944 B_j which is a sub-type of C . By induction hypothesis, for all j , all sub-terms of q_j satisfy
 945 the sub-type property for $\Gamma, x_j : B_j; A$. So all sub-terms of $\overline{\lambda x.q}$ satisfy the sub-type
 946 property for $\Gamma; A$ and we are done.

947

948 ► **Theorem 72.** *In λ^C , given a context Γ and a type D , the problem $\Gamma \vdash? : D$ is decidable.*
 949 *That is, it is whether there is a term t for which $\Gamma \vdash t : D$.*

950 **Proof.** By Theorem 69 we can limit our search to a term in normal form. So we can restrict
 951 the elimination rules to the following restricted case, where $\Phi = c(A_1, \dots, A_n)$. (Compare
 952 with the original rules in Definition 27.)

$$\frac{x : \Phi \in \Gamma \quad \dots \Gamma \vdash p_k : A_k \quad \dots \Gamma, y_\ell : A_\ell \vdash q_\ell : D}{\Gamma \vdash x \cdot_r [\bar{p}; \overline{\lambda y.q}] : D} \text{el}$$

953 Now, given Γ and D , the following algorithm searches a term t in normal form with
 954 $\Gamma \vdash t : D$. (1) Check if $x : D \in \Gamma$ for some x and otherwise (2) try an introduction rule (in
 955 case D is composite) and (3) try an elimination rule for each $x : \Phi \in \Gamma$ with Φ a composite
 956 formula. In the recursive case, this gives finitely many possibilities to try and each try creates
 957 new goals of the form $\Gamma, y_j : A_j \vdash? : D$ or of the form $\Gamma \vdash? : A_i$ with A_j and A_i sub-formulas
 958 of Γ, D . This search terminates because the number of sub-formulas in the context increases
 959 (which is bound by the number of all sub-formulas of Γ, D), and otherwise the size of the
 960 goal-formula decreases.

961

962 As a corollary, we find that all the variants of the logical rules we have considered are
 963 decidable and consistent, simply because they are (with respect to derivability) equivalent to
 964 the set of rules for $\wedge, \vee, \rightarrow, \neg, \perp, \top$ that we extract from the truth tables, for which Theorems
 965 70 and 72 apply. We can also say a bit more about the conversion of derivations in these
 966 systems themselves: detour conversion is strongly normalizing, permutation conversion is
 967 strongly normalizing and we can also conclude weak normalization of the combined conversion.

968 ► **Theorem 73.** *The reductions for the optimized rules of Definition 36, the parallel \wedge -*
 969 *elimination rule of Definition 44, the standard \rightarrow -introduction of Definition 47 and the*
 970 *traditional rule for \neg -elimination of Definition 50 are weakly normalizing.*

971 **Proof.** The proof follows the same argument as the proof of Theorem 69. The crucial Lemmas
 972 are Lemmas 68 and 66, which can be proved again with the reduction rules mentioned in the
 973 statement of Theorem 73 added. Furthermore, the permutation conversion, \rightarrow_b is strongly
 974 normalizing. (Corollary 56.)

975 **7 Conclusion and Further work**

976 We have studied the general procedure for deriving intuitionistic natural deduction rules from
 977 truth tables, that we have presented in [7]. We have defined detour conversion and permutation
 978 in general and we have proven that both are strongly normalizing and that the combination
 979 of the two is weakly normalizing. We have done so by defining a proof-term calculus for
 980 derivations on which we have defined the reduction rules that correspond to conversion of
 981 derivations. This follows the well-known Curry-Howard formulas-as-types isomorphism that
 982 establishes an isomorphism between proofs (derivations in natural deduction) and terms. We
 983 have shown that very many well-known formalisms for intuitionistic natural deduction can
 984 be defined in terms of our calculus, including the conversion rules for derivations. Our paper
 985 also provides a straightforward method for deriving a term calculus for any connective that
 986 is given via a truth table: the term constructions and reduction rules are self-contained and
 987 normalizing by construction. We have shown this on various examples, most notably the
 988 **nand**-connective.

989 The work described here leaves various questions unanswered. For example, is proof
 990 normalization (the combination of detour conversion and permutation conversion) strongly
 991 normalizing in general for an arbitrary set of connectives? We would believe so, but have not
 992 yet proved it. Techniques as in [9], where this property is proved for intuitionistic logic, may
 993 be useful.

994 It also raises various new research questions: The rules are not Church-Rosser (confluent)
 995 in general, but one may wonder whether there is a certain condition that guarantees confluence.
 996 We have seen in Examples 23, 30 and 40 that fixing a choice for the “matching case” in a
 997 detour convertibility may render the reduction confluent. It is not clear if this would work in
 998 general.

999 Another topic to look into is detour conversion for the classical case, and what its
 1000 connection is with known term calculi for classical logic, for example as studied in [13], [1]
 1001 and [2]. Also, it might be interesting to look at these general rules from a linear perspective:
 1002 what if we enforce the rules to be linear?

1003 Finally, we may wonder whether our research could contribute to the study of “harmony
 1004 in logic”, as first introduced by Prawitz [15] and further studied by various authors like
 1005 [16, 12, 23, 4, 3]. The inversion principle explains the elimination rules as capturing the
 1006 “least information” that is conveyed by the introduction rules. This can also be dualized (as
 1007 is done in [12] in their “uniform calculus”) by explaining the introduction rules in terms of
 1008 the elimination rules. It would be interesting to study the relation with our rules, where
 1009 there is no a priori preference for the introduction or elimination rules.

1010 From our research, we would propose the following as a proper system for intuitionistic
 1011 logic with “parallel elimination rules” that follow Prawitz’ [15] inversion principle. These rules
 1012 are derived from the truth tables and optimized following Lemma 9, but not using Lemma
 1013 12. Compare with Definition 13; the special rules are \wedge -elimination and \rightarrow -elimination.

► **Definition 74.** The *parallel elimination rules* for the intuitionistic propositional connectives

$\wedge, \vee, \rightarrow, \neg, \perp$ and \top are given below.

$\frac{\vdash A \quad \vdash B}{\vdash A \wedge B} \wedge\text{-in}$	$\frac{\vdash A \wedge B \quad A \vdash D}{\vdash D} \wedge\text{-el}_0$	$\frac{\vdash A \wedge B \quad B \vdash D}{\vdash D} \wedge\text{-el}_{-0}$
$\frac{\vdash A}{\vdash A \vee B} \vee\text{-inl}$	$\frac{\vdash B}{\vdash A \vee B} \vee\text{-inr}$	$\frac{\vdash A \vee B \quad A \vdash D \quad B \vdash D}{\vdash D} \vee\text{-el}$
$\frac{A \vdash A \rightarrow B}{\vdash A \rightarrow B} \rightarrow\text{-in}_a$	$\frac{\vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}_b$	$\frac{\vdash A \rightarrow B \quad \vdash A \quad B \vdash D}{\vdash D} \rightarrow\text{-el}$
$\frac{A \vdash \neg A}{\vdash \neg A} \neg\text{-in}$	$\frac{\vdash \neg A \quad \vdash A}{\vdash D} \neg\text{-el}$	$\frac{}{\vdash \top} \top\text{-in} \qquad \frac{\vdash \perp}{\vdash D} \perp\text{-el}$

1014 ——— References ———

- 1015 **1** Z. Ariola and H. Herbelin. Minimal classical logic and control operators. In *ICALP*, volume
1016 2719 of *LNCS*, pages 871–885. Springer, 2003.
- 1017 **2** P.-L. Curien and H. Herbelin. The duality of computation. In *ICFP*, pages 233–243, 2000.
- 1018 **3** R. Dyckhoff. Some remarks on proof-theoretic semantics. In *Advances in Proof-Theoretic*
1019 *Semantics*, pages 79–93. Springer, 2016.
- 1020 **4** N. Francez and R. Dyckhoff. A note on harmony. *Journal of Philosophical Logic*,
1021 41(3):613–628, 2012. URL: <http://dx.doi.org/10.1007/s10992-011-9208-0>, doi:10.
1022 1007/s10992-011-9208-0.
- 1023 **5** R.O. Gandy. An early proof of normalization by A.M. Turing. In J.P. Seldin and J.R.
1024 Hindley, editors, *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and*
1025 *Formalism*, page 453–455. Academic Press Limited, 1980.
- 1026 **6** G. Gentzen. Untersuchungen über das logische Schliessen. *Mathematische Zeitschrift*, pages
1027 176–210, 405–431, 1935. English translation in [19].
- 1028 **7** H. Geuvers and T. Hurkens. Deriving natural deduction rules from truth tables. In *ICLA*,
1029 volume 10119 of *Lecture Notes in Computer Science*, pages 123–138. Springer, 2017.
- 1030 **8** J.-Y. Girard et al. *Proofs and types*, volume 7 of *Cambridge tracts in theoretical computer*
1031 *science*. Cambridge University Press, Cambridge, 1989.
- 1032 **9** F. Joachimski and R. Matthes. Short proofs of normalization for the simply- typed lambda-
1033 calculus, permutative conversions and Gödel’s T. *Arch. Math. Log.*, 42(1):59–87, 2003.
- 1034 **10** E.G.K López-Escobar. Standardizing the N systems of Gentzen. In *Models, Algebras*
1035 *and Proofs*, volume 203 of *Lecture Notes in Pure and Applied Mathematics*, page 411–434.
1036 Marcel Dekker Inc., New York, 1999.
- 1037 **11** P. Milne. Inversion principles and introduction rules. In *Dag Prawitz on Proofs and Mean-*
1038 *ing*, volume 7 of *Outstanding Contributions to Logic*, pages 189–224. Springer, 2015.
- 1039 **12** S. Negri and J. von Plato. *Structural Proof Theory*. Cambridge University Press, 2001.
- 1040 **13** M. Parigot. $\lambda\mu$ -calculus: An algorithmic interpretation of classical natural deduction. In
1041 *LPAR*, volume 624 of *LNCS*, pages 190–201. Springer, 1992.
- 1042 **14** D. Prawitz. *Natural deduction: a proof-theoretical study*. Almqvist & Wiksell, 1965.
- 1043 **15** D. Prawitz. Ideas and results in proof theory. In J. Fenstad, editor, *2nd Scandinavian Logic*
1044 *Symposium*, pages 237–309. North-Holland, 1971.
- 1045 **16** S. Read. Harmony and autonomy in classical logic. *J. Philosophical Logic*,
1046 29(2):123–154, 2000. URL: <https://doi.org/10.1023/A:1004787622057>, doi:10.1023/
1047 A:1004787622057.

- 1048 **17** P. Schroeder-Heister. A natural extension of natural deduction. *J. Symb. Log.*, 49(4):1284–
1049 1300, 1984.
- 1050 **18** Morten Heine Sørensen and Pawel Urzyczyn. *Lectures on the Curry-Howard Isomorphism,*
1051 *Volume 149 (Studies in Logic and the Foundations of Mathematics)*. Elsevier Science Inc.,
1052 New York, NY, USA, 2006.
- 1053 **19** M.E. Szabo. *The Collected Papers of Gerhard Gentzen*. North-Holland, Amsterdam, 1969.
- 1054 **20** W.W. Tait. Intensional interpretations of functionals of finite type I. *J. Symb.*
1055 *Log.*, 32(2):198–212, 1967. URL: <http://dx.doi.org/10.2307/2271658>, doi:10.2307/
1056 2271658.
- 1057 **21** N. Tennant. Ultimate normal forms for parallelized natural deductions. *Logic Journal of*
1058 *the IGPL*, 10(3):299–337, 2002.
- 1059 **22** D. van Dalen. *Logic and structure (3. ed.)*. Universitext. Springer, 1994.
- 1060 **23** J. von Plato. Natural deduction with general elimination rules. *Arch. Math. Log.*, 40(7):541–
1061 567, 2001.