MRI master class Computer Mathematics

For the course notes see `http://www.cs.kun.nl/ herman/lect2.ps.gz`

Exercises 2: Polymorphic λ-calculus

Recall the following definitions

$$
\begin{array}{rcl@{\quad}rcl}
\bot & := & \forall\alpha.\alpha & \neg\sigma & := & \sigma\to\bot \\
\sigma\wedge\tau & := & \forall\alpha.(\sigma\to\tau\to\alpha)\to\alpha & \sigma\vee\tau & := & \forall\alpha.(\sigma\to\alpha)\to(\tau\to\alpha)\to\alpha \\
\exists\alpha.\sigma & := & \forall\beta.(\forall\alpha.\sigma\to\beta)\to\beta & & &
\end{array}
$$

1. Find inhabitants of the following types (both in the Curry and the Church variant of $\lambda 2$).
   (a) $\forall\alpha\forall\beta.\beta\to\alpha\to\beta$
   (b) $\sigma\to\neg\neg\sigma$
   (c) $\neg(\alpha\to\beta)\to\beta\to\alpha$
2. Assign a type to the following terms in the Curry variant of $\lambda 2$.
   (a) $\lambda x.xx$.
   (b) $\lambda x.x(\lambda z.x)$.
3. One can show the derivability of the logical rules for the conectives as defined above. Find closed terms of the following types.
   (a) $\sigma\to\tau\to(\sigma\wedge\tau)$ ($\wedge$-intro)
   (b) $\sigma\to(\sigma\vee\tau)$ ($\vee$-intro)
   (c) $\forall\alpha.\sigma\to\neg\sigma\to\alpha$ ($\neg$-elim)
   (d) $\sigma[\tau/\alpha]\to\exists\alpha.\sigma$ ($\exists$-intro).
4. Define the following functions in $\lambda 2$. (The functions Plus and Mult, given in the course notes and can be taken for granted.)
   (a) The faculty function on the natural numbers: $\mathrm{Fac}:\mathrm{Nat}\to\mathrm{Nat}$ such that $\mathrm{Fac}\,0 = 1$ and $\mathrm{Fac}\,(S\,n) = (S\,n) * (\mathrm{Fac}\,n)$.
   (b) The SumList function that sums all elements of a list of numbers: $\mathrm{SumList} : \mathrm{List}_{\mathrm{Nat}}\to\mathrm{Nat}$ such that $\mathrm{SumList}\,\mathrm{Nil} = 0$ and $\mathrm{SumList}\,(\mathrm{Cons}\,n\,l) = n + (\mathrm{SumList}\,l)$.
5. Prove the Soundness of $[\![-]\!]_\rho$ in the proof of Strong Normalization (the Proposition on slide 19):

$$x_1:\tau_1,\ldots,x_n:\tau_n \vdash M:\sigma \Rightarrow M[P_1/x_1,\ldots,P_n/x_n] \in [\![\sigma]\!]_\rho$$

   for all valuations ρ and $P_1 \in [\![\tau_1]\!]_\rho,\ldots,P_n \in [\![\tau_n]\!]_\rho$