MRI master class Computer Mathematics

For the course notes see `http://www.cs.kun.nl/ herman/lect4.ps.gz`

Exercises 4: Higher Order Logic

1. Define the Leibniz equality on A as $t =_A q := \forall P{:}A{\to}\mathsf{Prop}.(P\,t){\to}(P\,q)$. Prove the following by finding terms of the associated types.

   (a) reflexivity of $=_A$: $\forall x{:}A.x =_A x$.

   (b) transitivity of $=_A$: $\forall x, y, z{:}A.x =_A y \to y =_A z \to x =_A z$.

   (c) symmetry of $=_A$: $\forall x, y{:}A.x =_A y \to y =_A x$.

2. On the slides, the transitive closure of a relation R is defined as follows.

   $\mathsf{trclos} := \lambda R{:}A{\to}A{\to}\mathsf{Prop}.\lambda x, y{:}A.(\forall Q{:}A{\to}A{\to}\mathsf{Prop}.(\mathsf{trans}(Q){\to}(R \subseteq Q){\to}(Q\,x\,y)))$.

   So trclos is of type $(A{\to}A{\to}\mathsf{Prop}){\to}(A{\to}A{\to}\mathsf{Prop})$

   (a) Define the notions trans and $\subseteq$ in the definition of trclos.

   (b) Prove that the transitive closure is transitive. (Find a term of type $\mathsf{trans}(\mathsf{trclos}\,R)$).

   (c) Prove that the transitive closure of R contains R. (Find a term of type $R \subseteq (\mathsf{trclos}\,R)$).

3. Given a domain $A : \mathsf{Set}$, we identify $A{\to}\mathsf{Prop}$ with the collection of subsets of A. In this exercise we consider maps $\Phi : (A{\to}\mathsf{Prop}){\to}(A{\to}\mathsf{Prop})$, mapping subsets of A to subsets of A.
   Φ is called *monotone* if $\forall P, Q{:}A{\to}\mathsf{Prop}.P \subseteq Q \to (\Phi\,P) \subseteq (\Phi\,Q)$.
   ($P \subseteq Q$ is an abbreviation for $\forall x{:}A.(P\,x){\to}(Q\,x)$, which is also the answer to the exercise above.)
   $P : A{\to}\mathsf{Prop}$ is called Φ-*closed* if $(\Phi\,P) \subseteq P$.

   (a) Define (formally) $X : A{\to}\mathsf{Prop}$ as the *smallest* Φ-*closed* subset of A.

   (b) Prove (for arbitrary $P : A{\to}\mathsf{Prop}$): if P is Φ-closed, then $X \subseteq P$.
   (Find a term of type $\forall P{:}A{\to}\mathsf{Prop}.(\Phi\,P) \subseteq P \to X \subseteq P$.)

   (c) Prove $(\Phi\,X) \subseteq X$.

   (d) Prove $X \subseteq (\Phi\,X)$.

   (e) Conclude that X is the *least fixed point of* Φ:

      i. $X \approx (\Phi\,X)$,

      ii. $P \approx (\Phi\,P) \to X \subseteq P$.

   where we take the equality $\approx$ to be defined in the set-theoretical way as
   $P \approx Q := P \subseteq Q \wedge Q \subseteq P$.

4. Recall the induction principle over natural numbers as a higher order formula. Given $N : \mathsf{Set}, 0 : N, S : N \rightarrow N$, Ind_N is

$$\forall P{:}N \rightarrow \mathsf{Prop}.(P\,0) \rightarrow (\forall x{:}N.(P\,x) \rightarrow (P(S\,x))) \rightarrow \forall x{:}N.(P\,x)$$

   (a) Consider a datatype of lists over a base type A. So declare $A : \mathsf{Set}, L : \mathsf{Set}, \mathrm{Nil} : L, \mathrm{Cons} : A \rightarrow L \rightarrow L$. Define the induction principle over lists.

   (b) Consider a datatype of binary trees with leaves in base type A and node labels in base type B. So declare $A, B : \mathsf{Set}, T : \mathsf{Set}, \mathrm{Leaf} : A \rightarrow T, \mathrm{Join} : B \rightarrow T \rightarrow T \rightarrow T$. Define the induction principle over these trees.