Huygens College Reflection

Assignment 6, Tuesday, Jan. 5, 2016

Exercise 1

(i) Given is the data type `Nat` with `z` : `Nat`, `s` : `Nat` → `Nat`.
Write down the codes (following Böhm, Guerrini, Piperno) of

$$2 = s(sz),\ 3 = s(s(sz)).$$

(ii) Predecessor on `Nat` can be defined recursively:

$$\begin{aligned} p(0) &= 0 \\ p(n+1) &= n. \end{aligned}$$

Using the theory you learned, construct a term P of the form $\langle\langle B_1, B_2\rangle\rangle$, to act on codes of `Nat`, such that

$$\begin{aligned} P(\ulcorner z \urcorner) &= \ulcorner 0 \urcorner \\ P(\ulcorner sn \urcorner) &= \ulcorner n \urcorner. \end{aligned}$$

(iii) Verify $P\ulcorner 3\urcorner = \ulcorner 2\urcorner$.

Solution:

(i) The codes of $2 = s(sz)$ and $3 = s(s(sz))$ are defined by
$\ulcorner 2\urcorner = \ulcorner s(sz)\urcorner = \lambda e.e\mathbf{U}^2_2\ulcorner sz\urcorner e$
$\ulcorner sz\urcorner = \lambda f.f\mathbf{U}^2_2\ulcorner z\urcorner f$
$\ulcorner z\urcorner = \lambda g.g\mathbf{U}^2_1 g$ and
$\ulcorner 3\urcorner = \ulcorner s(s(sz))\urcorner = \lambda e.e\mathbf{U}^2_2\ulcorner s(sz)\urcorner e$
Filling in gives the precise codes:
$\ulcorner 2\urcorner = \lambda e.e\mathbf{U}^2_2(\lambda f.f\mathbf{U}^2_2(\lambda g.g\mathbf{U}^2_1 g)f)e$
$\ulcorner 3\urcorner = \lambda h.h\mathbf{U}^2_2(\lambda e.e\mathbf{U}^2_2(\lambda f.f\mathbf{U}^2_2(\lambda g.g\mathbf{U}^2_1 g)f)e)h.$

(ii) We want

$$\begin{aligned} P(\ulcorner z\urcorner) &= \ulcorner 0\urcorner = A_1 P \\ P(\ulcorner sn\urcorner) &= \ulcorner n\urcorner = A_2\ulcorner n\urcorner P \end{aligned}$$

This is the case if $A_1 = K\ulcorner 0\urcorner$ en $A_2 = K$.
Choose $B_1 = \lambda z.A_1\langle z\rangle$ and $B_2 = \lambda tz.A_2 t\langle z\rangle$, then the following works

$$P = \langle\langle B_1, B_2\rangle\rangle = \langle\langle \lambda z.\mathbf{K}\ulcorner 0\urcorner z, \lambda tz.\mathbf{K}t\langle z\rangle\rangle\rangle = \langle\langle \lambda z.\ulcorner 0\urcorner, \lambda tz.t\rangle\rangle = \langle\langle \mathbf{K}\ulcorner 0\urcorner, \mathbf{K}\rangle\rangle.$$

(iii)
$$\begin{aligned} P\ulcorner 3\urcorner &= \langle\langle \lambda xy.\ulcorner 0\urcorner, \mathbf{K}\rangle\rangle\ulcorner s(s(sz))\urcorner &&= \ulcorner s(s(sz))\urcorner\langle \lambda xy.\ulcorner 0\urcorner, \mathbf{K}\rangle \\ &= \langle \lambda xy.\ulcorner 0\urcorner, \mathbf{K}\rangle\mathbf{U}^2_2\ulcorner s(sz)\urcorner\langle \lambda xy.\ulcorner 0\urcorner, \mathbf{K}\rangle &&= \ulcorner s(sz)\urcorner \\ &= \ulcorner 2\urcorner. \end{aligned}$$

Exercise 2

(i) Given is `Tree`, the data type with

$$\texttt{l} : \texttt{Tree}, \texttt{j} : \texttt{Tree}^2 \to \texttt{Tree}.$$

Write down the codes (following BGP) of

$$t_1 = j\,(j\,l\,l)\,l \quad \text{and} \quad t_2 = j\,l\,(j\,l\,l).$$

(ii) Write down a λ-term $F = \langle\langle D_1, D_2 \rangle\rangle$ (to act on codes of `Tree`) such that

$$\begin{array}{rcl} F\ulcorner l \urcorner & = & l \\ F\ulcorner j\,t\,s \urcorner & = & \ulcorner j\,t\,(j\,t\,s)\urcorner. \end{array}$$

(iii) Verify for the F you found that indeed $F\ulcorner j\,l\,(j\,l\,l)\urcorner = \ulcorner j\,l\,(j\,l\,(j\,l\,l))\urcorner$.

Solution:

(i)
$$\begin{array}{rcl} \ulcorner t_1 \urcorner & = & \ulcorner j(jll)l \urcorner = \lambda e_1.e_1 \mathbf{U}_2^2 \ulcorner jll \urcorner \ulcorner l \urcorner e_1 \\ \ulcorner jll \urcorner & = & \lambda e_2.e_2 \mathbf{U}_2^2 \ulcorner l \urcorner \ulcorner l \urcorner e_2 \\ \ulcorner l \urcorner & = & \lambda e_3.e_3 \mathbf{U}_1^2 e_3 \\ \\ \ulcorner t_2 \urcorner & = & \ulcorner jl(jll) \urcorner = \lambda e_1.e_1 \mathbf{U}_2^2 \ulcorner l \urcorner \ulcorner jll \urcorner e_1. \end{array}$$

Filling in gives

$$\ulcorner t_1 \urcorner = \lambda e_1.e_1 \mathbf{U}_2^2 (\lambda e_2.e_2 \mathbf{U}_2^2 (\lambda e_3.e_3 \mathbf{U}_1^2 e_3)(\lambda e_3.e_3 \mathbf{U}_1^2 e_3) e_2)(\lambda e_3.e_3 \mathbf{U}_1^2 e_3) e_1$$

$$\ulcorner t_2 \urcorner = \lambda e_1.e_1 \mathbf{U}_2^2 (\lambda e_3.e_3 \mathbf{U}_1^2 e_3)((\lambda e_2.e_2 \mathbf{U}_2^2 (\lambda e_3.e_3 \mathbf{U}_1^2 e_3)(\lambda e_3.e_3 \mathbf{U}_1^2 e_3) e_2)) e_1.$$

(ii) We are looking for an F with

$$\begin{array}{c} F\ulcorner l \urcorner = \ulcorner l \urcorner \\ F\ulcorner jts \urcorner = \ulcorner jt(jts) \urcorner \end{array}$$

Following the BPG-coding we try $F \triangleq \langle\langle D_1, D_2 \rangle\rangle$. We want $D_1 \langle D_1, D_2 \rangle = \ulcorner l \urcorner$. Which is what we get in case $D_1 = \lambda x.\ulcorner l \urcorner$.

Furthermore we want

$$\begin{array}{rcl} D_2 \ulcorner t \urcorner \ulcorner s \urcorner \langle D_1, D_2 \rangle & = & \ulcorner jt(jts) \urcorner \\ & \equiv & \lambda e.e \mathbf{U}_2^2 \ulcorner t \urcorner \ulcorner jts \urcorner e \\ & \equiv & \lambda e.e \mathbf{U}_2^2 \ulcorner t \urcorner (\lambda f.f \mathbf{U}_2^2 \ulcorner t \urcorner \ulcorner s \urcorner f) e. \end{array}$$

This we get if we take $D_2 = \lambda xyz.\lambda e.e \mathbf{U}_2^2 x (\lambda f.f \mathbf{U}_2^2 xyf) e$.

(iii) Now we have
$$\begin{array}{rcl} F\ulcorner jl(jll) \urcorner & = & \ulcorner jl(jll) \urcorner \langle D_1, D_2 \rangle \\ & = & D_2 \ulcorner l \urcorner \ulcorner jll \urcorner \langle D_1, D_2 \rangle \\ & = & \lambda e.e \mathbf{U}_2^2 \ulcorner l \urcorner (\lambda f.f \mathbf{U}_2^2 \ulcorner l \urcorner \ulcorner jll \urcorner f) e \\ & = & \lambda e.e \mathbf{U}_2^2 \ulcorner l \urcorner e \mathbf{U}_2^2 \ulcorner l \urcorner \ulcorner jll \urcorner e \\ & = & \lambda e.e \mathbf{U}_2^2 \ulcorner l \urcorner \ulcorner jl(jll) \urcorner e \\ & = & \ulcorner jl(jl(jll)) \urcorner. \end{array}$$

Exercise 3

Check the statement on Slide 10 of the course slides, that

$$\begin{array}{rcl} H(\mathtt{Var}\,x) & =_\beta & A_1\,x\,H \\ H(\mathtt{App}\,x\,y) & =_\beta & A_2\,x\,y\,H \\ H(\mathtt{Abs}\,x) & =_\beta & A_3\,x\,H \end{array}$$

if we take $H = \langle\langle B_1, B_2, B_3\rangle\rangle$ with

$$\begin{array}{rcl} B_1 & := & \lambda x\,z.A_1 x\langle z\rangle \\ B_2 & := & \lambda x\,y\,z.A_2 xy\langle z\rangle \\ B_3 & := & \lambda x\,z.A_3 x\langle z\rangle. \end{array}$$

and `Var`, `App` and `Abs` as on the slides. (Verify 2 of the equations for H.)

Solution: Fill in the values for the B_i and do the β-equalities

Exercise 4

Show that there is no term F such that

$$F(M\,N) =_\beta N \text{ for all terms } M, N.$$

Solution: Suppose $F(M\,N) =_\beta N$ for all M, N. Then $F(\mathbf{IK}) = \mathbf{K}$ but also $F(\mathbf{IK}) = F(\mathbf{K}) = F(\mathbf{KKI}) = \mathbf{I}$. So $\mathbf{K} = \mathbf{I}$, which is a contradiction.

Exercise 5

Remember the definitions of **true** $:= \lambda xy.x (\equiv \mathbf{K})$ and **false** $= \lambda xy.y (=_\beta \mathbf{KI})$.

(i) Construct a λ-term G such that

$$\begin{array}{rcl} G\ulcorner x\urcorner & = & \mathbf{true} \\ G\ulcorner PQ\urcorner & = & \mathbf{false} \\ G\ulcorner \lambda x.P\urcorner & = & \mathbf{false}. \end{array}$$

(ii) Construct a λ-term V such that

$$\begin{array}{rcl} V\ulcorner x\urcorner & = & \mathbf{true} \\ V\ulcorner PQ\urcorner & = & V\ulcorner P\urcorner \\ V\ulcorner \lambda x.P\urcorner & = & \mathbf{false}. \end{array}$$

Solution:

(i) Try $G = \langle\langle B_1, B_2, B_3\rangle\rangle$ for some B_1, B_2, B_3. Then
$G\ulcorner x\urcorner = \ulcorner x\urcorner\langle B_1, B_2, B_3\rangle = \langle B_1, B_2, B_3\rangle U_1^3 x\langle B_1, B_2, B_3\rangle = B_1 x\langle B_1, B_2, B_3\rangle$, so take $B_1 = \lambda xy.$**true**
$G\ulcorner PQ\urcorner = \ulcorner PQ\urcorner\langle B_1, B_2, B_3\rangle = \langle B_1, B_2, B_3\rangle U_2^3\ulcorner P\urcorner\ulcorner Q\urcorner\langle B_1, B_2, B_3\rangle = B_2\ulcorner P\urcorner\ulcorner Q\urcorner\langle B_1, B_2, B_3\rangle$. So take $B_2 = \lambda xyz.$**false**.

$G\ulcorner\lambda x.P\urcorner = \langle B_1, B_2, B_3\rangle U_3^3(\lambda x.\ulcorner P\urcorner)\langle B_1, B_2, B_3\rangle = B_3(\lambda x.\ulcorner P\urcorner)\langle B_1, B_2, B_3\rangle$.
So $B_3 = \lambda xy.\mathbf{false}$.

Conclusion: $G = \langle\langle \lambda xy.\mathbf{true}, \lambda xyz.\mathbf{false}, \lambda xy.\mathbf{false}\rangle\rangle$

(ii) Because V and G are the same for $\ulcorner x\urcorner$ and $\ulcorner\lambda x.M\urcorner$, their B_1 and B_3 are also the same.

$V\ulcorner PQ\urcorner = B_2\ulcorner P\urcorner\ulcorner Q\urcorner\langle B_1, B_2, B_3\rangle = V\ulcorner P\urcorner = \langle\langle B_1, B_2, B_3\rangle\rangle\ulcorner P\urcorner = \ulcorner P\urcorner\langle B_1, B_2, B_3\rangle$.
So $B_2 = \lambda xyz.xz$.

Conclusion: $V = \langle\langle \lambda xy.\mathbf{true}, \lambda xyz.xz, \lambda xy.\mathbf{false}\rangle\rangle$.