Deterministic Finite Automata

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Outline

Finite Automata

Manipulating finite automata

Finite automata and regular languages

Regular languages
Definition

$L \subseteq A^*$ is regular if $L = \mathcal{L}(e)$ for some regular expression e.

Decidability Problems

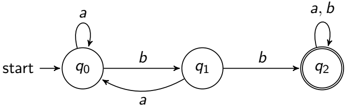
- Q1 Can we decide (algorithmically) if $w \in \mathcal{L}(e)$?
- Q2 If yes, what is the complexity of this decision procedure?
- Q3 Can we decide if $\mathcal{L}(e_1) = \mathcal{L}(e_2)$?

Some answers

- A1 Yes, we can give a deterministic finite automaton for e to decide $w \in \mathcal{L}(e)$. (This lecture)
- A2 time is linear in $|w|$; memory is constant (independent of w).
- A3 Yes: $\mathcal{L}(e_1) = \mathcal{L}(e_2)$ can be axiomatised, see Exercise 2.3.6 of the course notes. It is decidable whether $e_1 = e_2$ according to these axioms.

Deterministic Finite State Automaton (DFA)

Intuition. Let $A = \{a, b\}$. Consider the DFA M:



Letters a, b are the moves in the graph
A $w \in A^*$ is a sequence of moves
Start state is indicated by 'start$\rightarrow$',
Accepting states by double circle
(there can be several accepting states)

The word *abba* is accepted, but *baab* is not accepted (rejected)
$M = (Q, q_0, \delta, F)$ with $Q = \{q_0, q_1, q_2\}$, $F = \{q_2\}$ and δ given by

δ	a	b
q_0	q_0	q_1
q_1	q_0	q_2
q_2	q_2	q_2

Deterministic Finite Automata formally

M is a DFA over A if $M = (Q, q_0, \delta, F)$ with

A	is a finite alphabet
Q	is a finite set of states
$q_0 \in Q$	is the initial state
$F \subseteq Q$	is a finite set of final states
$\delta : Q \times A \to Q$	is the transition function

The multi-step transition $\delta^* : Q \times A^* \to Q$ is defined inductively by

$$\begin{aligned} \delta^*(q, \lambda) &= q \\ \delta^*(q, aw) &= \delta^*(\delta(q, a), w) \end{aligned}$$

The language accepted at state q, notation $\mathcal{L}(q)$, is:

$$\mathcal{L}(q) = \{w \in A^* \mid \delta^*(q, w) \in F\}$$

The language accepted by M, notation $\mathcal{L}(M)$, is $\mathcal{L}(q_0)$.

Reading words $w \in A^*$

Computation for $\delta^*(q_0, w)$ in the example DFA. Take $w = abba$:

$$
\begin{array}{rcll}
[q_0, abba] & \vdash & [\delta(q_0, a), bba] & = [q_0, bba] \\
 & \vdash & [\delta(q_0, b), ba] & = [q_1, ba] \\
 & \vdash & [\delta(q_1, b), a] & = [q_2, a] \\
 & \vdash & [\delta(q_2, a), \lambda] & = [q_2, \lambda]
\end{array}
$$

$$
\begin{array}{rcll}
[q_0, aba] & \vdash & [\delta(q_0, a), ba] & = [q_0, ba] \\
 & \vdash & [\delta(q_0, b), a] & = [q_1, a] \\
 & \vdash & [\delta(q_1, a), \lambda] & = [q_0, \lambda]
\end{array}
$$

So *abba* is acepted and *aba* is not accepted.
The language accepted by M (of the first slide) is regular. It is the language

$$\mathcal{L}((a+b)^* bb(a+b)^*).$$

From transition table to state diagram

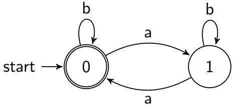
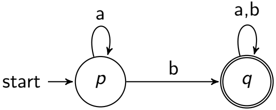
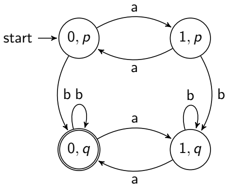
Consider the automaton M over $A = \{a, b\}$ with

- $Q = \{0, 1, 2, 3, 4\}$,
- $q_0 = 0$,
- $F = \{4\}$
-

δ	a	b
0	1	0
1	1	2
2	1	3
3	4	0
4	4	4

1. Which of the following words is accepted? *abba*, *baba*, *bba*
2. Is it the case that $\{w \mid |w|_b \text{ is even } \} \subseteq \mathcal{L}(M)$?
3. Is it the case that $\{w \mid w \text{ contains } aabbaa\} \subseteq \mathcal{L}(M)$?

Manipulating Finite Automata: products for intersection

M	$\mathcal{L}(M)$
	$L_1 = \{w \mid \lvert w\rvert_a \text{ is even}\}$
	$L_2 = \{w \mid \lvert w\rvert_b \geq 1\}$
	$L_1 \cap L_2 =$ $\{w \mid \lvert w\rvert_a \text{ is even and } \lvert w\rvert_b \geq 1\}$

Product of two DFAs

Given two DFAs over the same A

$$\begin{aligned} M_1 &= (Q_1, q_{01}, \delta_1, F_1) \\ M_2 &= (Q_2, q_{02}, \delta_2, F_2) \end{aligned}$$

Define

$$M_1 \times M_2 = (Q_1 \times Q_2, q_0, \delta, F)$$

with

$$\begin{aligned} q_0 &:= (q_{01}, q_{02}) \\ \delta((q_1, q_2), a) &:= (\delta_1(q_1, a), \delta_2(q_2, a)) \end{aligned}$$

Then with

$$F := F_1 \times F_2 := \{(q_1, q_2) \mid q_1 \in F_1 \text{ and } q_2 \in F_2\}$$

we have

$$\mathcal{L}(M_1 \times M_2) = \mathcal{L}(M_1) \cap \mathcal{L}(M_2)$$

Closure Properties

Proposition Closure under complement

If L is accepted by some DFA, then so is

$$\overline{L} = A^* - L.$$

Proof. Suppose that L is accepted by $M = (Q, q_0, \delta, F)$.
Then $\overline{L}$ is accepted by $M = (Q, q_0, \delta, \overline{F})$. ☻

Proposition Closure under intersection and union

If L_1, and L_2 are accepted by some DFA, then so are $L_1 \cap L_2$ and $L_1 \cup L_2$.

Proof. For the intersection, this follows from the product construction on the previous slide.

For the union, this can be seen by the product construction, taking a different F (which one?) or by noticing that $L_1 \cup L_2 = \overline{\overline{L_1} \cap \overline{L_2}}$.

☻

Kleene's Theorem

Theorem The languages accepted by DFAs are exactly the regular languages

We will prove this in this and the next lecture by

1. If $L = \mathcal{L}(M)$, for some DFA M, then there is a regular expression e such that $L = \mathcal{L}(e)$ (this lecture).
2. If $L = \mathcal{L}(e)$, for some regular expression e, then there is a non-deterministic finite automaton (NFA) M such that $L = \mathcal{L}(M)$. (next lecture).
3. For every NFA M, there is a DFA M' such that $\mathcal{L}(M) = \mathcal{L}(M')$ (next lecture)

From DFAs to regular expressions

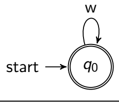
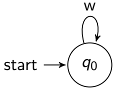
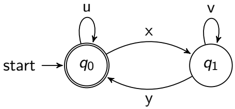
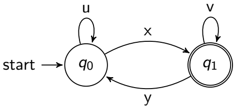
Given the DFA $M = (Q, q_0, \delta, F)$, we construct a regular expression e such that

$$\mathcal{L}(e) = \mathcal{L}(M).$$

Procedure:

- We remove states,
  replacing symbols from A by regular expressions over A,
- until we end up with a "simple automaton" from which we can read off e.

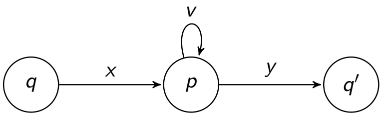
Simple automata

M	e such that $\mathcal{L}(e) = \mathcal{L}(M)$
	w^*
	0
	$(u + xv^*y)^*$
	$u^*x(v + yu^*x)^*$

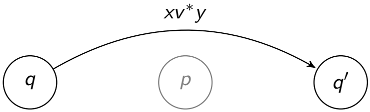
Eliminating states

- Remove a state p,
- while adding arrows $q \xrightarrow{w} q'$ between other pairs of states.

Before:

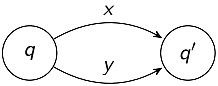


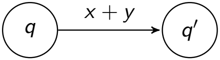
After:



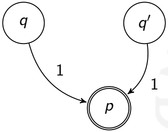
Special cases

Join arrows using $+$

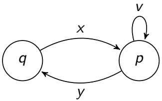


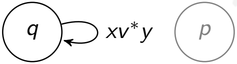


First create a new single final state



Beware of loops!





From DFA to RegExp

The algorithm outlined in the previous slides produces for every deterministic finite automaton M, a regular expression e_M.

Theorem For M a DFA we have

$$\mathcal{L}(e_M) = \mathcal{L}(M).$$