Non-regular languages

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Outline

The Class of Regular Languages

The Pumping Lemma for Regular Languages

Organisational

Next week is different:

- Test-exam on part I (lectures 1-3)
- Make sure you are registered for the course in Blackboard by Friday Dec. 4!
- Time: 15:45 – 17:30, make sure you are there at 15:35.
- Locations: LIN3 and LIN6

Equivalence of definitions

Theorem. Let $L \subseteq \Sigma^*$. Then the following are equivalent

1. L is "**machine-regular**", i.e. $L = \mathcal{L}(M)$ for some DFA (or NFA, NFA_λ)
2. L is **regular**, i.e. $L = \mathcal{L}(e)$ for some regular expression e.

(Proof. See previous lectures.)

So:

- To show that a language is regular we can give a regular expression or a (non-)deterministic automaton (with λ-steps).
- To show closure properties of the class of regular languages, we can use regular expressions, deterministic automata, non-deterministic automata, ...

Closure properties of the Class of regular languages

If L, L_1 and L_2 over Σ are regular then so are

- $\overline{L}$ (NB. $\overline{L} = \{w \in \Sigma^* \mid w \notin L\}$)
- L^*
- $L_1 \cup L_2$
- $L_1 \cap L_2$
- $L_1 L_2$
- L^R (NB. $L^R = \{w \in \Sigma^* \mid w^R \in L\}$)
- Prefix(L)
  NB. $\text{Prefix}(L) := \{w \in \Sigma^* \mid \exists v \in L(w \text{ is a prefix of } v)\}$
  w is a prefix of v if $v = wu$ for some $u \in \Sigma^*$.

Example of a language that is *not* regular

Lemma The language

$$L := \{a^n b^n \in \Sigma^* \mid n \geq 0\}$$

is not regular

How to prove this? Showing that there is no regular expression that describes L? Showing that there is no DFA (NFA, NFA$_\lambda$) that accepts L?

Proof Suppose the DFA M accepts L.
Then for all $n, p \in \mathbb{N}$, if $n \neq p$, then $\delta^*(q_0, a^n) \neq \delta^*(q_0, a^p)$.
Why?
Because, if $\delta^*(q_0, a^n) = \delta^*(q_0, a^p) = q$, then $\delta^*(q, b^p) \in F$ (final state), but then $a^n b^p$ is also accepted, while it shouldn't be.
So M must have infinitely many states, which is not the case.
So there is no DFA M accepting L. ☻

Non regular languages

Let $\Sigma = \{a, b\}$. We will develop a general technique that can be used to show that languages are not regular.
This technique will be applied to show that

$$\{a^n b^n \in \Sigma^* \mid n \geq 0\}$$

is not regular
and to show that

$$\{w \in \Sigma^* \mid w \text{ is a palindrome}\}$$

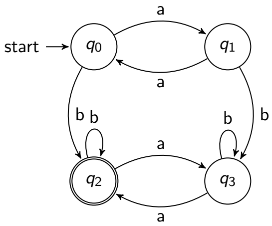
is not regular.
A *palindrome* is a word w such that $w^R = w$.
Remember that w^R is the *reverse of* w, defined by

$$\begin{aligned} \lambda^R &:= \lambda \\ (s\,w)^R &:= w^R\,s \end{aligned}$$

A general method to show that a language is *not* regular

Regular languages can be **pumped**!

Example: Consider $\Sigma = \{a, b\}$ and the automaton



accepting

$$\{w \in \Sigma^* \mid |w|_b \geq 1 \wedge |w|_a \text{ is even}\}$$

What happens if a word of length $4, 5, 6, 7, \ldots$ is accepted?

It has made a cycle which can be repeated arbitrarily often!

For example, *baaaa* is accepted, and also all $baa(aa)^n$ are accepted.

We say that *aa* is a substring that can be pumped.

Pumping Lemma for Regular Languages

Pumping Lemma. Let $L \subseteq \Sigma^*$ be a regular language
Then there exists a number $k \geq 1$ (pumping number) such that
for every $w \in L$ with $|w| \geq k$ one has the following

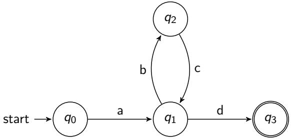
1. w can be split in three parts, $w = uvz$,
2. with $|uv| \leq k$ and $|v| \geq 1$,
3. such that for all $n \geq 0$ one has $uv^n z \in L$.

Corollary $L = \{a^n b^n \mid n \geq 0\}$ **is not regular**
Proof. Suppose L is regular. (Towards a contradiction.)
Let $k \geq 1$ be as in the Pumping Lemma
Take $w = a^k b^k$. Then $w \in L$ and $|w| \geq k$
Therefore there are u, v, z such that we can write $a^k b^k = uvz$,
with $|uv| \leq k$, $|v| \geq 1$ and $uv^n z \in L$ for all $n \geq 0$.
Then $v = a^q$, for some $q \geq 1$. But then $uv^2 z = a^{k+q} b^k \notin L$.
Contradiction. ☺

Proof of the Pumping Lemma

Let L be regular. Let M be a DFA that accepts L.
Take k to be the number of states of M.
Let $w \in L$ with $|w| \geq k$. Then, reading word w, we must pass some state more than once.
Say that q is the first state that we pass twice (reading w).
Then $w = uvz$, where we read u to go to q, read v to loop at q, read z to go to a final node.
But then $uv^n z$ is accepted for all n. ☻

Example $abcd \in \mathcal{L}(M)$ because of the following path in M:



Since q_1 is visited twice we can pump:
$a(bc)^n d \in \mathcal{L}(M)$ for all $n \geq 0$.

Negating formulas

$$\begin{array}{rcl}
\neg\exists x.P(x) & \iff & \forall x.\neg P(x) \\
\neg\forall x.P(x) & \iff & \exists x.\neg P(x) \\
\neg\exists x.[Q(x) \wedge P(x)] & \iff & \forall x.[Q(x) \Rightarrow \neg P(x)] \\
\neg\forall x.[Q(x) \Rightarrow P(x)] & \iff & \exists x.[Q(x) \wedge \neg P(x)]
\end{array}$$

Using the Pumping Lemma to prove non-regularity

Pumping lemma. L is regular $\Rightarrow$ L can be pumped
We use this as follows:

$$L \text{ cannot be pumped } \Rightarrow L \text{ is not regular}$$

L can be pumped means:
$\exists k \geq 1 \forall w \in L.(|w| \geq k \Rightarrow$
$\exists u, v, z\, [w = uvz \wedge |uv| \leq k \wedge |v| \geq 1 \wedge \forall n \in \mathbb{N}\, (uv^n z \in L)])$
L cannot be pumped means:
$\forall k \geq 1 \exists w \in L.(|w| \geq k \wedge$
$\forall u, v, z\, [w = uvz \wedge |uv| \leq k \wedge |v| \geq 1 \Rightarrow \exists n \in \mathbb{N}\, (uv^n z \notin L)])$
To show that L is **not regular** it suffices to show it cannot be pumped.

How to prove non-regularity using the pumping lemma

To show that L is not regular we do the following:

For each $k \geq 1$, **find some** $w \in L$ of length $\geq k$ so that

- **for every** way of splitting up w as $w = u\,v\,z$,
- with $|u\,v| \leq k$ and $|v| \geq 1$,
- you can **find an** $n \geq 0$ for which $u\,v^n\,z$ is not in L.

Application: $L = \{w \in \Sigma^* \mid w \text{ is a palindrome}\}$ is not regular.
Proof. We follow the procedure above.
Let $k \geq 1$ (arbitrary)
Take $w = a^k b a^k$. Then $w \in L$ (**check**) and $|w| \geq k$ (**check**)
Let u, v, z (arbitrary) be so that $a^k b a^k = uvz$, with $|uv| \leq k$ and $|v| \geq 1$. (Say $|v| = p \geq 1$.)
Take $n = 0$. Then $uv^n z = uv^0 z = a^{k-p} b a^k \notin L$ (**check**).
So, L is not regular. ☻

Some other non-regularity results

Let $\Sigma := \{a, b\}$. We know that $L = \{a^n b^n \mid n \geq 0\}$ is not regular.
Is $L' := \{w \in \Sigma^* \mid \forall n \in \mathbb{N}\, (w \neq a^n b^n)\}$ regular?

Answer: No it is not. If L' is regular, then $\overline{L'}$ would also be regular, but this is just L and L is not regular! So L' is not regular.

Lemma If L is *not* regular, then also $\overline{L}$ and L^R are not regular

Let $\Sigma := \{a, b, c\}$.
Is $L'' := \{a^n c^p b^n \in \Sigma^* \mid n \geq 0, p \geq 0\}$ regular?

Answer: No it is not. $L = L'' \cap \mathcal{L}(a^* b^*)$. If L'' is regular, then L would be regular as well, but it is not!

Lemma If L is *not* regular and $L = L_1 \cap L_2$, with L_1 regular, then L_2 is not regular.

In proving non-regularity:

You may use

- The Pumping Lemma
- The fact that $\{a^n b^n \in \Sigma^* \mid n \geq 0\}$ is not regular.
- The fact that $\{w \in \Sigma^* \mid w \text{ is a palindrome}\}$ is not regular.
- Closure properties of the class of regular languages.