# Non-deterministic Finite Automata

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# Outline

# Previous Weeks

## Regular Expressions and Regular Languages

$$\mathtt{rexp}_\Sigma ::= 0 \mid 1 \mid s \mid \mathtt{rexp}_\Sigma\ \mathtt{rexp}_\Sigma \mid \mathtt{rexp}_\Sigma + \mathtt{rexp}_\Sigma \mid \mathtt{rexp}_\Sigma^*$$

with $\mathtt{s} \in \Sigma$
$L \subseteq \Sigma^*$ is regular if $L = \mathcal{L}(e)$ for some regular expression $e$.

## Deterministic Finite Automata, DFA

**Proposition** Closure under complement, union, intersection
If $L_1, L_2$ are accepted by some DFA, then so are

- $\overline{L_1} = \Sigma^* - L_1$
- $L_1 \cup L_2$
- $L_1 \cap L_2$.

# Kleene's Theorem (announced last lecture)

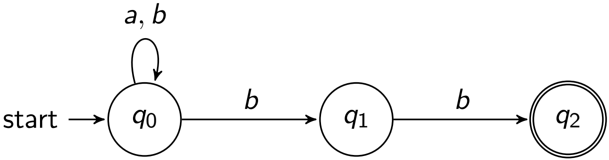
**Theorem**

The languages accepted by DFAs are exactly the regular languages

We prove this by

1. If $L = \mathcal{L}(M)$ for some DFA $M$, then there is a regular expression $e$ such that $L = \mathcal{L}(e)$ (Previous lecture)
2. If $L = \mathcal{L}(e)$, for some regular expression $e$, then there is a non-deterministic finite automaton with $\lambda$-steps (NFA-$\lambda$) $M$ such that $L = \mathcal{L}(M)$. (This lecture)
3. For every NFA-$\lambda$, $M$, there is a DFA $M'$ such that $\mathcal{L}(M) = \mathcal{L}(M')$ (This lecture)

# Non-deterministic finite automaton (NFA)




$\delta(q, a)$ is not one state, but a set of states.

| $\delta$ | $a$ | $b$ |
|---|---|---|
| $q_0$ | $\{q_0\}$ | $\{q_0, q_1\}$ |
| $q_1$ | $\emptyset$ | $\{q_2\}$ |
| $q_2$ | $\emptyset$ | $\emptyset$ |

in shorthand

| $\delta$ | $a$ | $b$ |
|---|---|---|
| $q_0$ | $q_0$ | $q_0, q_1$ |
| $q_1$ | | $q_2$ |
| $q_2$ | | |

# Non-deterministic Finite Automata: NFA (definition)

$M$ is a NFA over $\Sigma$ if $M = (Q, q_0, \delta, F)$ with

| | |
|---|---|
| $Q$ | is a finite set of states |
| $q_0 \in Q$ | is the initial state |
| $F \subseteq Q$ | is a finite set of final states |
| $\delta : Q \times \Sigma \to \mathcal{P}Q$ | is the transition function |
| | [$\mathcal{P}Q$ denotes the collection of subsets of $Q$] |

Reading function $\delta^* : Q \times \Sigma^* \to \mathcal{P}Q$ (multi-step transition)

$$\begin{aligned} \delta^*(q, \lambda) &= \{q\} \\ \delta^*(q, aw) &= \{q' \mid q' \in \delta^*(p, w) \text{ for some } p \in \delta(q, a)\} \\ &= \bigcup_{p \in \delta(q,a)} \delta^*(p, w) \end{aligned}$$

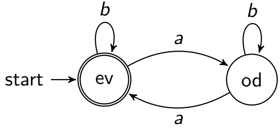
The language accepted by $M$, notation $\mathcal{L}(M)$, is:

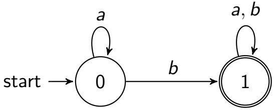
$$\mathcal{L}(M) = \{w \in \Sigma^* \mid \exists q_f \in \delta^*(q_0, w) \text{ such that } q_f \in F\}$$

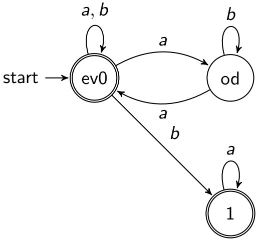
# For the union of languages we can put NFAs in parallel

Example Suppose we want to have an NFA for $L_1 \cup L_2 = \{w \mid |w|_a \text{ is even or } |w|_b \geq 1\}$

First idea: put the two machines "non-deterministically in parallel"



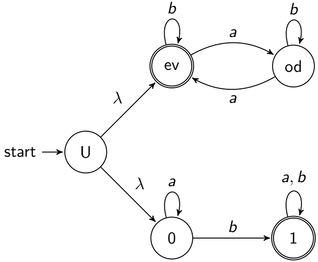




But this is wrong: The NFA accepts *aaa*.

# NFAs with silent steps: NFA-$\lambda$

We add $\lambda$-transitions or 'silent steps' to NFAs

The correct union of $M_1$ and $M_2$ is:




In an NFA-$\lambda$ we allow

$$\delta(q, \lambda) = q'$$

for $q \neq q'$. That means

$$\delta : Q \times (\Sigma \cup \{\lambda\}) \to \mathcal{P}Q$$

# NFA-$\lambda$ (definition)

$M$ is an NFA-$\lambda$ over $\Sigma$ if $M = (Q, q_0, \delta, F)$ with

| | |
|---|---|
| $Q$ | is a finite set of states |
| $q_0 \in Q$ | is the initial state |
| $F \subseteq Q$ | is a finite set of final states |
| $\delta : Q \times (\Sigma \cup \{\lambda\}) \to \mathcal{P}Q$ | is the transition function |

The $\lambda$-closure of a state $q$, $\lambda$-closure$(q)$, is the set of states reachable with only $\lambda$-steps.

Reading function $\delta^* : Q \times \Sigma^* \to \mathcal{P}Q$ (multi-step transition)

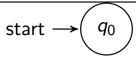
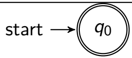
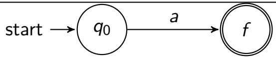
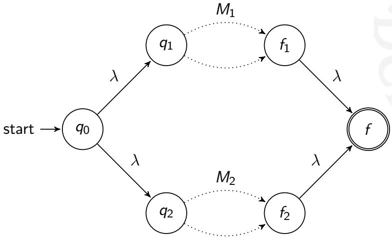
$$\begin{aligned}
\delta^*(q, \lambda) &= \lambda\text{-closure}(q) \\
\delta^*(q, aw) &= \{q' \mid \exists p \in \lambda\text{-closure}(q)\, \exists r \in \delta(p, a)\, (q' \in \delta^*(r, w))\} \\
&= \bigcup_{p \in \lambda\text{-closure}(q)} \; \bigcup_{r \in \delta(p,a)} \delta^*(r, w)
\end{aligned}$$

The language accepted by $M$, notation $\mathcal{L}(M)$, is:

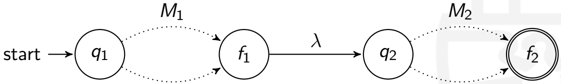
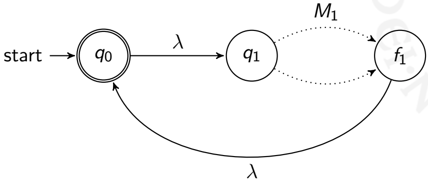
$$\mathcal{L}(M) = \{w \in \Sigma^* \mid \exists q_f \in \delta^*(q_0, w) \text{ such that } q_f \in F\}$$

# Toolkit for building an NFA-$\lambda$ from a regular expression

For each regular expression, we construct an NFA-$\lambda$.

| $e$ | $M$ such that $\mathcal{L}(M) = \mathcal{L}(e)$ |
|---|---|
| $0$ |  |
| $1$ |  |
| $a$ (for $a \in \Sigma$) |  |
| $e = e_1 + e_2$ with $\mathcal{L}(M_1) = \mathcal{L}(e_1)$ $\mathcal{L}(M_2) = \mathcal{L}(e_2)$ |  |

# Toolkit (continued)

| $e$ | $M$ such that $\mathcal{L}(M) = \mathcal{L}(e)$ |
|---|---|
| $e = e_1 e_2$ with $\mathcal{L}(M_1) = \mathcal{L}(e_1)$ $\mathcal{L}(M_2) = \mathcal{L}(e_2)$ |   |
| $e = (e_1)^*$ with $\mathcal{L}(M_1) = \mathcal{L}(e_1)$ |   |

# Regular languages accepted by a NFA-$\lambda$

Proposition. For every regular expression $e$ there is an NFA-$\lambda$ $M_e$ such that

$$\mathcal{L}(M_e) = \mathcal{L}(e).$$

Proof. Apply the toolkit. $M_e$ can be found *by induction on the structure of* $e$: First do this for the simplest regular expressions. For a composed regular expression compose the automata. ☻

Corollary. For every regular language $L$ there is an NFA-$\lambda$ $M$ that accepts $L$ (so $\mathcal{L}(M) = L$).

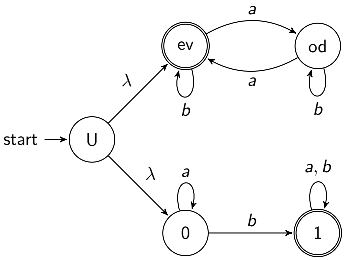
# Avoiding non-determinism

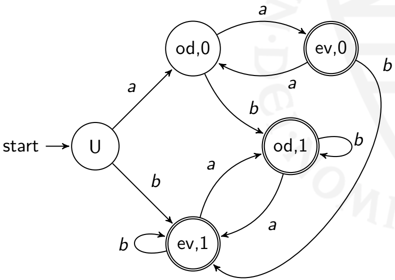
We can transform any NFA (and NFA-$\lambda$) into a DFA that accepts the same language.

Idea:

- Keep track of the set of all states you can go to!
- States of the DFA are sets-of-states from the original NFA-$\lambda$.
- A set of states is final if one of the members is final.

Example $L = \{w \mid |w|_a \text{ is even or } |w|_b \geq 1\}$



# Eliminating non-determinism and $\lambda$-steps

Let $M$ be a NFA given by $(Q, q_0, \delta, F)$
Define the DFA $\overline{M}$ as $(\overline{Q}, \overline{q_0}, \overline{\delta}, \overline{F})$ where

$$
\begin{aligned}
\overline{Q} &= \mathcal{P}Q \\
\overline{q_0} &= \{q_0\} \\
\overline{\delta}(H, a) &= \bigcup_{q \in H} \delta(q, a), \qquad \text{for } H \subseteq Q, \\
\overline{F} &= \{H \subseteq Q \mid H \cap F \neq \emptyset\}
\end{aligned}
$$

If $M$ is an NFA-$\lambda$, we define

$$
\begin{aligned}
\overline{q_0} &= \lambda\text{-closure}(q_0) \\
\overline{\delta}(H, a) &= \bigcup_{q \in H} \; \bigcup_{p \in \lambda\text{-closure}(q)} \lambda\text{-closure}(\delta(p, a)) \\
\overline{F} &= \{H \subseteq Q \mid \lambda\text{-closure}(H) \cap F \neq \emptyset\}
\end{aligned}
$$

# Correctness

Given $M$, an NFA-$\lambda$, we have defined the DFA $\overline{M}$ by

$$
\begin{aligned}
\overline{q_0} &= \{q_0\} \\
\overline{\delta}(H, a) &= \bigcup_{q \in H} \; \bigcup_{p \in \lambda\text{-closure}(q)} \lambda\text{-closure}(\delta(p, a)) \\
\overline{F} &= \{H \subseteq Q \mid \lambda\text{-closure}(H) \cap F \neq \emptyset\}
\end{aligned}
$$

Theorem $M$ and $\overline{M}$ accept the same languages.

Proof: This follows from
Lemma

$$\delta^*(q, w) \cap F \neq \emptyset \iff \overline{\delta}^*(\{q\}, w) \in \overline{F}$$

(Take $q := q_0$)

Proof of the Lemma: induction on $w$, considering the cases $w = \lambda$ and $w = au$.

# Equivalence of DFA, NFA and NFA-$\lambda$

Conclusion. Every NFA-$\lambda$ (or NFA) $M$ can be turned into a DFA $\overline{M}$ accepting the same language.

Corollary. For every regular language $L$ there is a DFA $M$ that accepts $L$ (so $\mathcal{L}(M) = L$).

Proof. Given a regular expression $e$, first construct an NFA-$\lambda$ $M$ such that $\mathcal{L}(M) = \mathcal{L}(e)$. Then change it into a *DFA* preserving the language that is accepted. ☻

Rephrasing of Kleene's Theorem:

The class of regular languages is (equivalently) characterized as

1. The languages described by a regular expression
2. The languages accepted by a DFA
3. The languages accepted by an NFA
4. The languages accepted by a NFA-$\lambda$