Non-regular languages

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Outline

Equivalence of definitions

Theorem. Let $L \subseteq \Sigma^*$. Then the following are equivalent

1. L is "**machine-regular**", i.e. $L = \mathcal{L}(M)$ for some DFA (or NFA, NFA-λ)
2. L is **regular**, i.e. $L = \mathcal{L}(e)$ for some regular expression e.

(Proof. See previous lectures.)

So:

- To show that a language is regular we can give a regular expression or a (non-)deterministic automaton (with λ-steps).
- To show closure properties of the class of regular languages, we can use regular expressions, deterministic automata, non-deterministic automata, ...

Closure properties of the Class of regular languages

If L, L_1 and L_2 over Σ are regular then so are

- $\overline{L}$ (NB. $\overline{L} = \{w \in \Sigma^* \mid w \notin L\}$)
- L^*
- $L_1 \cup L_2$
- $L_1 \cap L_2$
- $L_1 L_2$
- L^R (NB. $L^R = \{w \in \Sigma^* \mid w^R \in L\}$)
- Prefix(L)
  NB. Prefix(L) $:= \{w \in \Sigma^* \mid \exists v \in L(w \text{ is a prefix of } v)\}$
  w is a prefix of v if $v = wu$ for some $u \in \Sigma^*$.

Example of a language that is *not* regular

Lemma The language

$$L := \{a^n b^n \in \Sigma^* \mid n \geq 0\}$$

is not regular

How to prove this? Showing that there is no regular expression that describes L? Showing that there is no DFA (NFA, NFA-λ) that accepts L?

Proof Suppose the DFA $M = (Q, q_0, \delta, F)$ accepts L.
Then for all $n, m \in \mathbb{N}$, if $n \neq m$, then $\delta^*(q_0, a^n) \neq \delta^*(q_0, a^m)$.
[Why?
Because, if $\delta^*(q_0, a^n) = \delta^*(q_0, a^m) = q$, then $\delta^*(q, b^m) \in F$,
but then $a^n b^m$ is also accepted, while it shouldn't be.]
So M must have infinitely many states, which is not the case.
So there is no DFA accepting L, so L is not regular. ☺

Non regular languages

Let $\Sigma = \{a, b\}$. We will develop a general technique that can be used to show that languages are not regular.
This technique will be applied to show that

$$\{a^n b^n \in \Sigma^* \mid n \geq 0\}$$

is not regular
and to show that

$$\{w \in \Sigma^* \mid w \text{ is a palindrome}\}$$

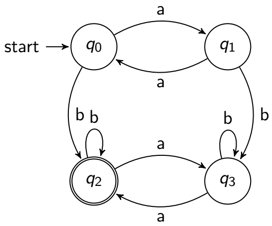
is not regular.
A *palindrome* is a word w such that $w^R = w$.
Remember that w^R is the *reverse of* w, defined by

$$\begin{array}{rcl} \lambda^R & := & \lambda \\ (s\,w)^R & := & w^R\,s \end{array}$$

A general method to show that a language is *not* regular

Regular languages can be **pumped**!

Example: Consider $\Sigma = \{a, b\}$ and the automaton



accepting

$$\{w \in \Sigma^* \mid |w|_b \geq 1 \wedge |w|_a \text{ is even}\}$$

What happens if a word of length $4, 5, 6, 7, \ldots$ is accepted?
It has made a cycle which can be repeated arbitrarily often!
For example, *baaaa* is accepted, and also all $baa(aa)^n$ are accepted.
We say that *aa* is a substring that can be pumped.

Pumping Lemma for Regular Languages

Pumping Lemma. Let $L \subseteq \Sigma^*$ be a regular language
Then there exists a number $k \geq 1$ (pumping number) such that
for every $w \in L$ with $|w| \geq k$ one has the following

1. w can be split in three parts, $w = uvz$,
2. with $|uv| \leq k$ and $|v| \geq 1$,
3. such that for all $n \geq 0$ one has $uv^n z \in L$.

Corollary $L = \{a^n b^n \mid n \geq 0\}$ **is not regular**
Proof. Suppose L is regular. (Towards a contradiction.)
Let $k \geq 1$ be as in the Pumping Lemma.
We take $w = a^k b^k$. Then $w \in L$ and $|w| \geq k$.
Therefore there are u, v, z such that $a^k b^k = uvz$, with $|uv| \leq k$,
$|v| \geq 1$ and $uv^n z \in L$ for all $n \geq 0$.
Then $v = a^q$, for some $q \geq 1$. We take $n = 2$ and conclude that
$uv^2 z = a^{k+q} b^k \in L \ldots$ But this wrong: $a^{k+q} b^k \notin L$!
Contradiction. So L is not regular. ☺

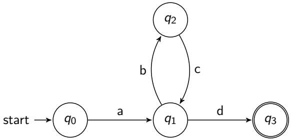
Proof of the Pumping Lemma

Pumping Lemma. If $L \subseteq \Sigma^*$ is regular, then
$\exists k \geq 1 \, \forall w \in L \, (|w| \geq k \Rightarrow$
$\exists u, v, z \, [w = uvz \wedge |uv| \leq k \wedge |v| \geq 1 \wedge \forall n \in \mathbb{N} \, (uv^n z \in L)])$

Proof. Let L be regular. Let M be a DFA that accepts L.
Take k to be the number of states of M.
Let $w \in L$ with $|w| \geq k$. Then, reading word w, we must pass some state more than once.
Say that q is the first state that we pass twice (reading w).
Then $w = uvz$, where we read u to go to q, read v to loop at q, read z to go to a final node.
Note that indeed $|uv| \leq k$ and $|v| \geq 1$. Then $uv^n z$ is accepted for all n. ☻

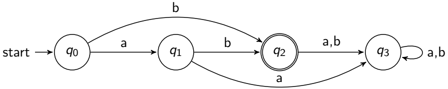
Examples of the Pumping Lemma

Example 1 Suppose $abcd \in \mathcal{L}(M_1)$ because of the following path:



Since q_1 occurs twice we can pump: $a(bc)^n d \in \mathcal{L}(M_1) \quad (\forall n \geq 0)$.

Example 2 What can we pump in the following DFA M_2? (What is the "pumping number"?)



For the "pumping number" we can take $k = 4$. Since $\mathcal{L}(M_2) = \{b, ab\}$ we indeed have $\forall w \in \mathcal{L}(M_2)\,(|w| \geq 4 \Rightarrow \ldots)$

Negating formulas

$$
\begin{aligned}
\neg\exists x.P(x) &\iff \forall x\,\neg P(x)\\
\neg\forall x\,P(x) &\iff \exists x\,\neg P(x)\\
\neg\exists x\,(Q(x)\wedge P(x)) &\iff \forall x\,(Q(x)\Rightarrow\neg P(x))\\
\neg\forall x\,(Q(x)\Rightarrow P(x)) &\iff \exists x\,(Q(x)\wedge\neg P(x))
\end{aligned}
$$

Using the Pumping Lemma to prove non-regularity

Pumping lemma. L is regular $\Rightarrow$ L can be pumped
We use this as follows:

$$L \text{ cannot be pumped } \Rightarrow L \text{ is not regular}$$

L can be pumped means:
$\exists k \geq 1 \forall w \in L\,(|w| \geq k \Rightarrow$
$\exists u, v, z\,[w = uvz \wedge |uv| \leq k \wedge |v| \geq 1 \wedge \forall n \in \mathbb{N}\,(uv^n z \in L)])$

L cannot be pumped means:
$\forall k \geq 1\,\exists w \in L\,(|w| \geq k \wedge$
$\forall u, v, z\,[w = uvz \wedge |uv| \leq k \wedge |v| \geq 1 \Rightarrow \exists n \in \mathbb{N}\,(uv^n z \notin L)])$

To show that L is **not regular** it suffices to show it cannot be pumped.

How to prove non-regularity using the pumping lemma

To show that L is not regular we do the following:

For each $k \geq 1$, **find some** $w \in L$ of length $\geq k$ so that

- **for every** way of splitting up w as $w = u\,v\,z$,
- with $|u\,v| \leq k$ and $|v| \geq 1$,
- you can **find an** $n \geq 0$ for which $u\,v^n\,z$ is not in L.

Application: $L = \{w \in \Sigma^* \mid w \text{ is a palindrome}\}$ is not regular.
Proof. We follow the procedure above.
Let $k \geq 1$ (arbitrary)
Take $w = a^k b a^k$. Then $w \in L$ (**check**) and $|w| \geq k$ (**check**)
Let u, v, z (arbitrary) be so that $a^k b a^k = uvz$, with $|uv| \leq k$ and $|v| \geq 1$. (Say $|v| = p$, so $p \geq 1$.)
Take $n = 0$. Then $uv^n z = uv^0 z = a^{k-p} b a^k \notin L$ (**check**).
So, L is not regular. ☻

Some other non-regularity results

Let $\Sigma := \{a, b\}$. We know that $L = \{a^n b^n \mid n \geq 0\}$ is not regular.
Is $L' := \{w \in \Sigma^* \mid \forall n \in \mathbb{N}\,(w \neq a^n b^n)\}$ regular?

Answer: No it is not. If L' is regular, then $\overline{L'}$ would also be regular, but this is just L and L is not regular! So L' is not regular.

Lemma If L is *not* regular, then also $\overline{L}$ and L^R are not regular

Let $\Sigma := \{a, b, c\}$.
Is $L'' := \{a^n c^p b^n \in \Sigma^* \mid n \geq 0, p \geq 0\}$ regular?

Answer: No it is not. $L = L'' \cap \mathcal{L}(a^* b^*)$. If L'' is regular, then L would be regular as well, but it is not!

Lemma If L is *not* regular and $L = L_1 \cap L_2$, with L_1 regular, then L_2 is not regular.

In proving non-regularity of a language L:

You may use

- The Pumping Lemma, in either one of the following ways:
  1. Assume that L is regular, so it satisfies the Pumping Lemma, and derive a contradiction.
     (Example proof of non-regularity of $\{a^n b^n \in \Sigma^* \mid n \geq 0\}$ on slide 9.)
  2. Show that L "cannot be pumped".
     (Example proof of non-regularity of $\{w \in \Sigma^* \mid w \text{ is a palindrome}\}$ on slide 14.)
- The fact that $\{a^n b^n \in \Sigma^* \mid n \geq 0\}$ is not regular.
- The fact that $\{w \in \Sigma^* \mid w \text{ is a palindrome}\}$ is not regular.
- Closure properties of the class of regular languages.