Regular Languages & Non-deterministic finite Automata

Regular Expressions and Regular Languages

$$\boxed{\texttt{rexp}_\Sigma ::= \emptyset \mid \lambda \mid s \mid \texttt{rexp}_\Sigma\ \texttt{rexp}_\Sigma \mid \texttt{rexp}_\Sigma \cup \texttt{rexp}_\Sigma \mid \texttt{rexp}_\Sigma^*}$$

with $\mathsf{s} \in \Sigma$

Deterministic Finite Automata, DFA

Proposition Closure under complement, union, intersection

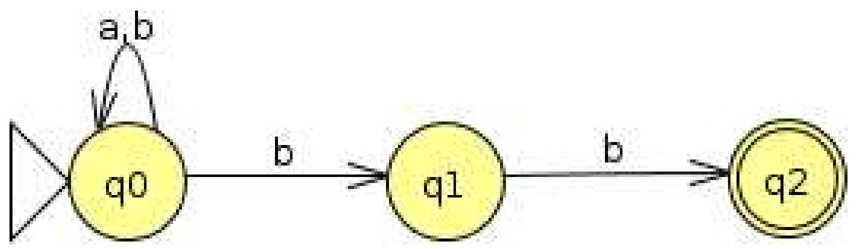
If L_1, L_2 are accepted by some DFA, then so are

- $\overline{L_1} = \Sigma^* - L_1$
- $L_1 \cup L_2$
- $L_1 \cap L_2$.

Theorem The languages accepted by DFAs are exactly the regular languages

We prove this by

1. If $L = L(M)$, for some DFA M, then there is a regular expression e such that $L = L(e)$ (previous lecture)
2. If $L = L(e)$, for some regular expression e, then there is a non-deterministic finite automaton with λ-steps (NFA$_\lambda$) M such that $L = L(M)$. (this lecture)
3. For every NFA$_\lambda$, M, there is a DFA M' such that $L(M) = L(M')$ (this lecture)



δ	q_0	q_1	$\boldsymbol{q}_2$
a	q_0	$\emptyset$	$\emptyset$
b	$\{q_0, q_1\}$	$\boldsymbol{q}_2$	$\emptyset$

in shorthand

δ	q_0	q_1	$\boldsymbol{q}_2$
a	q_0		
b	q_0, q_1	$\boldsymbol{q}_2$	

M is a DFA over Σ if $M = (Q, \Sigma, q_0, \delta, F)$ with

Q	is a finite set of states
Σ	is a finite alphabet
$q_0 \in Q$	is the initial state
$F \subseteq Q$	is a finite set of final states
$\delta : Q \times \Sigma \to \mathcal{P}(Q)$	is the transition function

$\mathcal{P}(Q)$ denotes the collection of subsets of Q

Reading function $\hat{\delta} : Q \times \Sigma^* \to \mathcal{P}(Q)$ (multi-step transition)

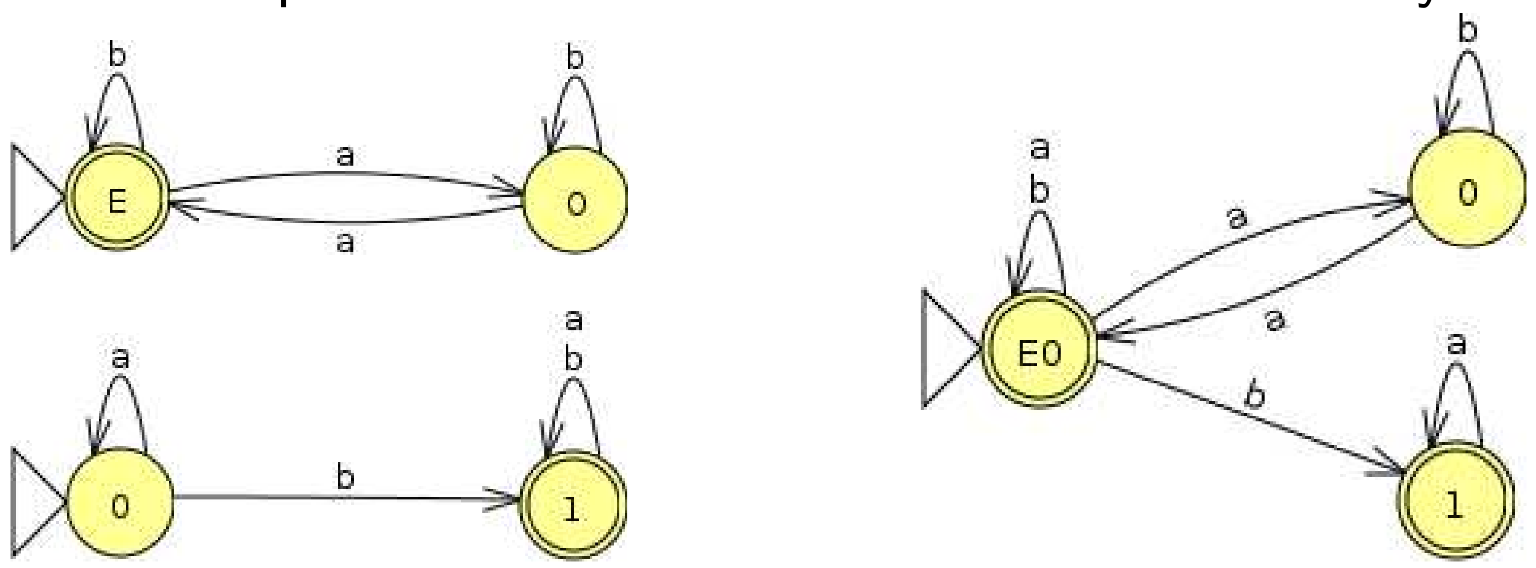
$$\begin{aligned} \hat{\delta}(q, \lambda) &= \{q\} \\ \hat{\delta}(q, wa) &= \{q'' \mid q'' \in \delta(q', a) \text{ for some } q' \in \hat{\delta}(q, w)\} \end{aligned}$$

The language accepted by M, notation $L(M)$, is:

$$L(M) = \{w \in \Sigma^* \mid \exists q_f \in F (q_f \in \hat{\delta}(q_0, w))\}$$

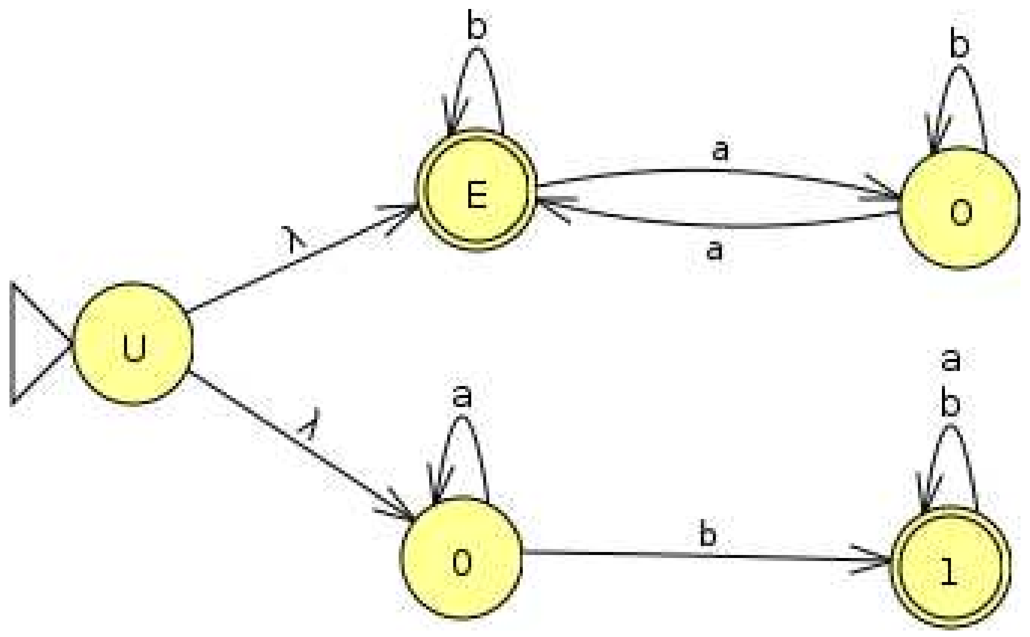
Suppose we want $\{w \mid \#_a \text{ even or } \#_b \geq 1\} = L_1 \cup L_2$

First idea: put the two machines "nondeterministically" together



The NFA on the right accepts 'aaa' which is wrong!

Now we add λ transitions or 'silent steps' to NFAs



In a NFA$_\lambda$ we allow

$$\delta(q, \lambda) = q'$$

for $q \neq q'$. That means

$$\delta : Q \times (\Sigma \cup \{\lambda\}) \to \mathcal{P}(Q)$$

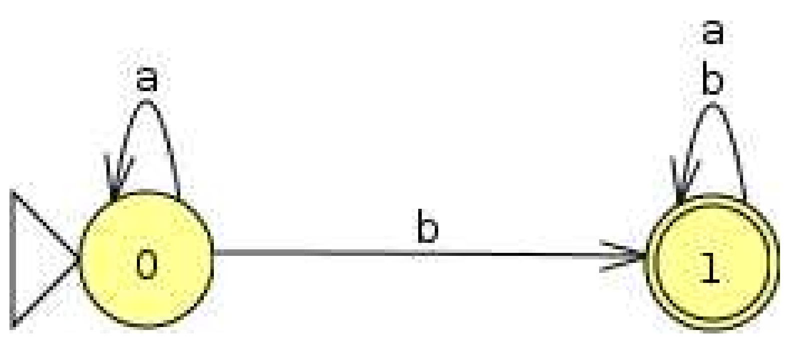
A finite automaton M is called insulated

(i) if q_0 has no in-going arrows
(ii) there is only one final state which has no out-going arrows

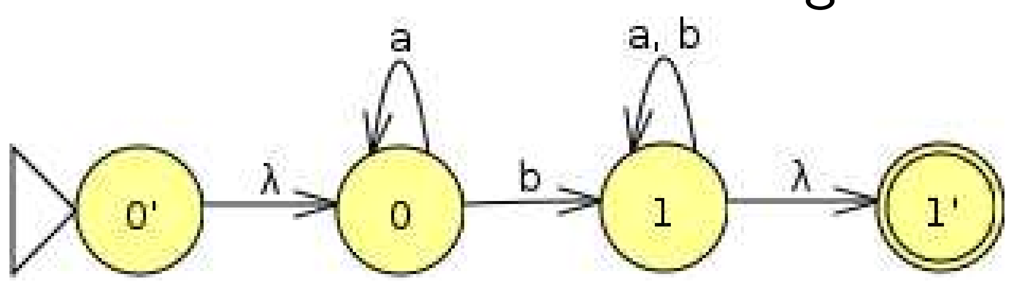
Proposition. One can insulate any machine M such that

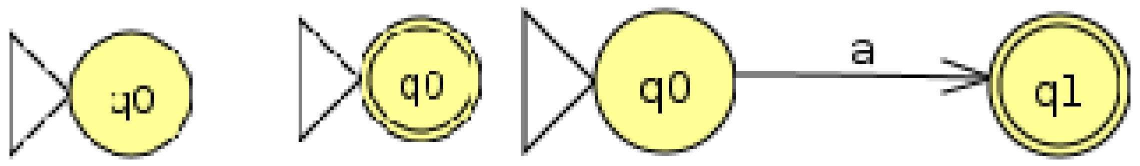
the result M' accepts the same language

Proof. By adding states and silent steps, for example



gives

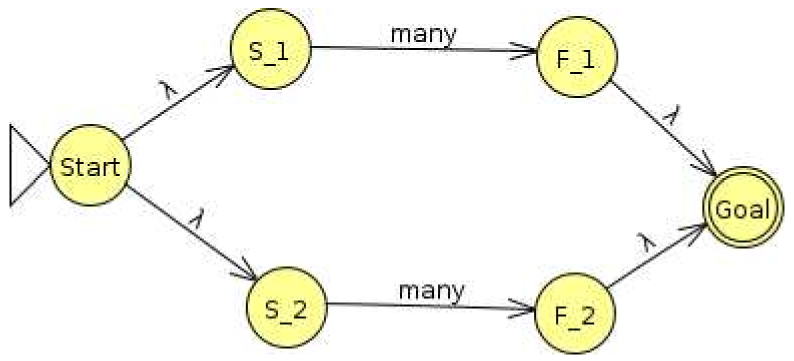




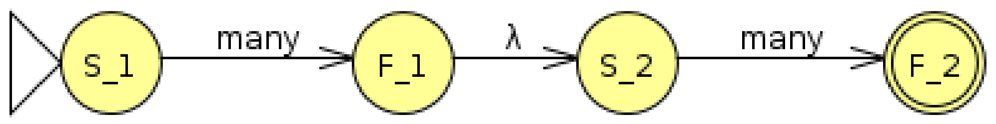
$\emptyset$ $\{\lambda\}$ $\{a\}$

$L_1 = L$

L_2



$L_1 \cup L_2$ insulated



L_1L_2 insulated

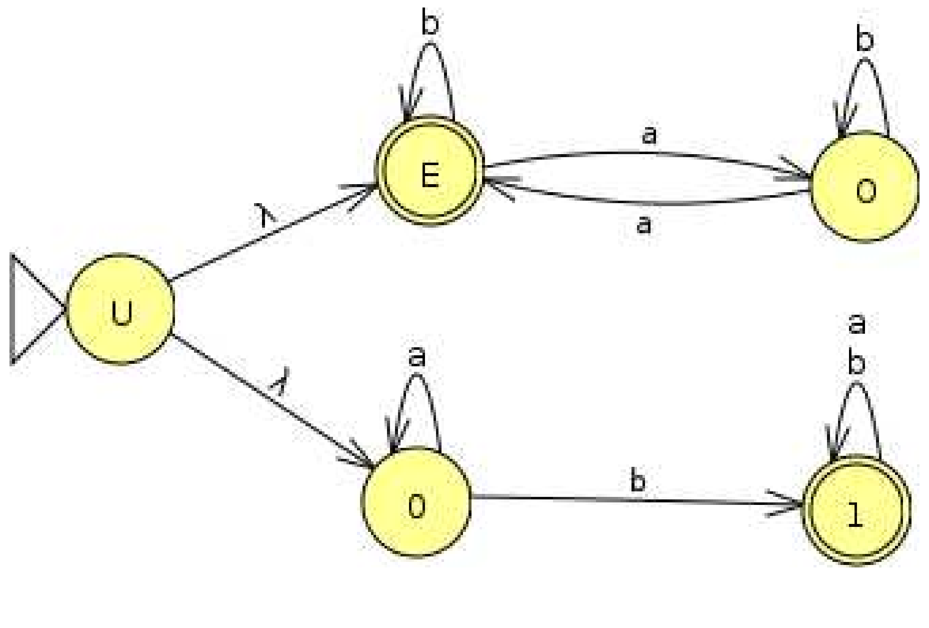
L^* insulated

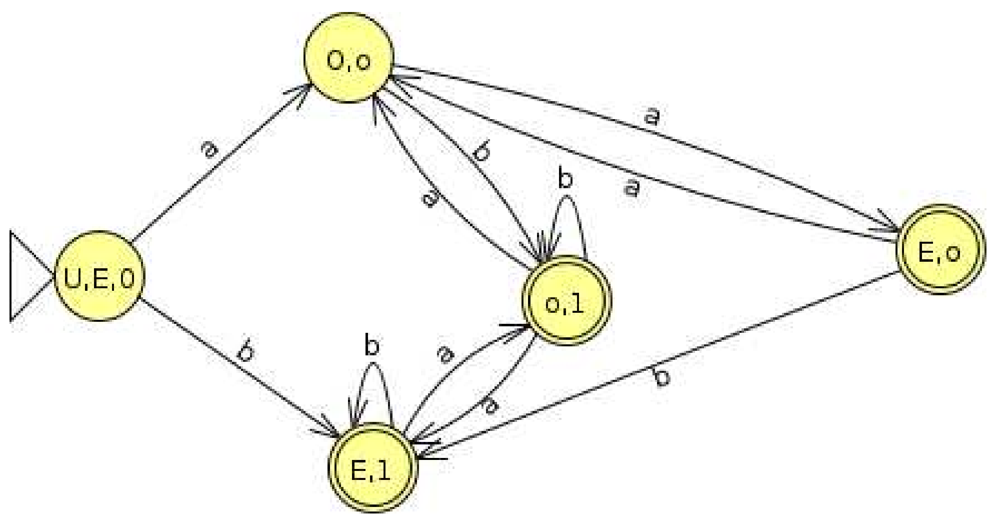
Proposition. For every regular expression e there is an NFA$_\lambda$ M_e such that

$$L(M_e) = L(e).$$

Proof. Apply the toolkit. M_e can be found 'by induction on the structure of e': first do this for the simplest regular expressions; then for a composed regular expression compose the automata. ■

Corollary. For every regular language L there is an NFA$_\lambda$ M that accepts L (so $L(M) = L$).





Keep track of where you can go!
A combination is final if one of the members is final.

We show how a NFA can be turned into a DFA

Let M be a NFA given by $(Q, \Sigma, q_0, \delta, F)$

Define M^+ as $(Q^+, \Sigma, q_0^+, \delta^+, F^+)$ by

$$\begin{aligned} Q^+ &= \mathcal{P}(Q) \\ q_0 &= \{q_0\} \\ \delta^+(H, a) &= \bigcup_{q \in H} \delta(q, a), \qquad \text{for } H \subseteq Q, \\ F^+ &= \{H \subseteq Q \mid H \cap F \neq \emptyset\} \end{aligned}$$

Then M^+ is a DFA accepting the same language as M

Every NFA_λ M can be turned into an NFA M' accepting the same language.

Corollary. For every regular language L there is a DFA M that accepts L (so $L(M) = L$).