# Context-free Languages & Pushdown Automata

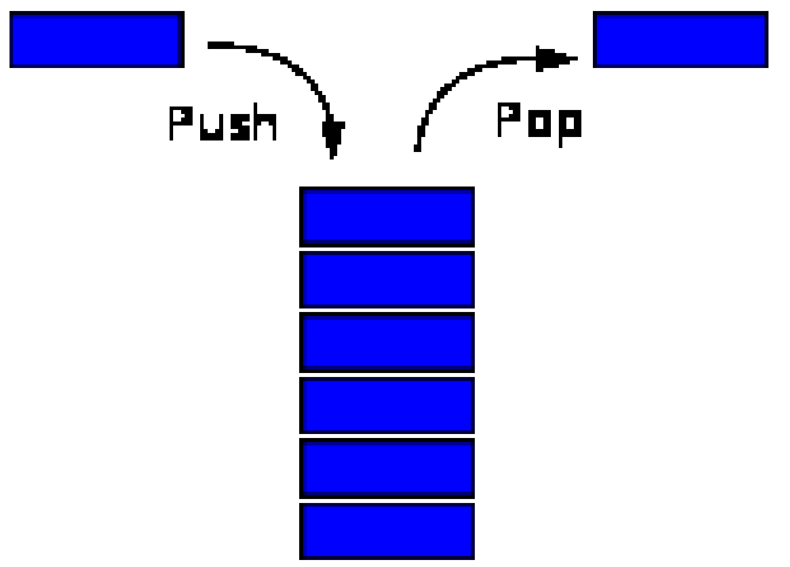
The limitation of DFAs is that they don't have a memory.

Various (simple) memory models are possible:

Queue First in, first out (like a plastic cup dispenser in a coffee machine)

Stack Last in, first out (like plates in a student restaurant)

Pushdown automata extend automata with a stack




Each item carries an element of *stack alphabet* $\Gamma$

*A pushdown automaton is a sextuple* $\langle Q, \Sigma, \Gamma, \delta, q_0, F\rangle$ *with*

$Q$ *a finite set of states*

$q_0$ *an element of* $Q$*, the initial state*

$F$ *a subset of* $Q$

$\Sigma$ *a finite set of symbols (input alphabet)*

$\Gamma$ *the* stack alphabet

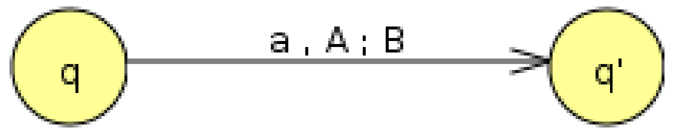
$\delta$ *a map ('afbeelding')*

$\delta : Q \times (\Sigma \cup \{\lambda\}) \times (\Gamma \cup \{\lambda\}) \rightarrow \mathcal{P}(Q \times \Gamma \cup \{\lambda\})$

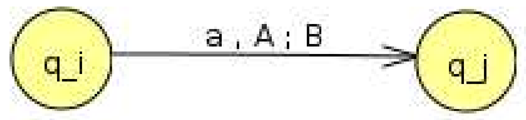
*We write e.g.* $\delta(q_i, a, A) = \{[q_j, B], [q_k, C]\}$

*We understand the* $Q, \Sigma, \mathcal{P}, \lambda$*. New is the* $\Gamma$*: alfabet of stack symbols*

*The stack is not mentioned, but it is used in the operation of the PDA!*



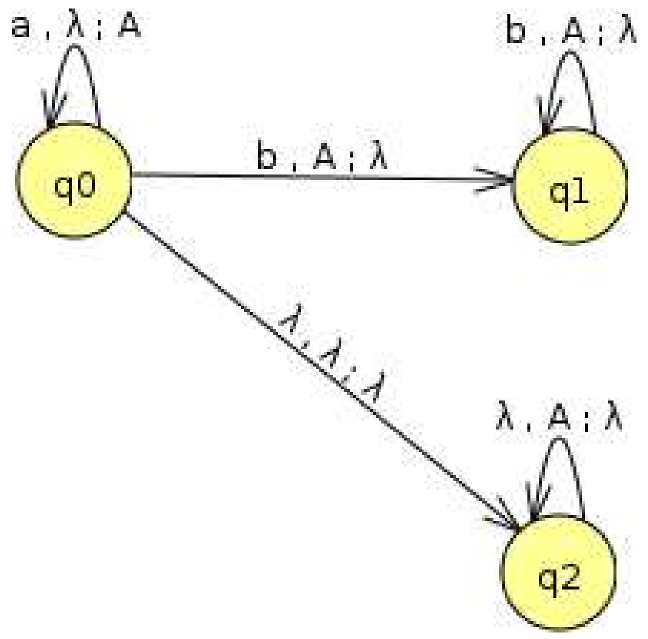

$[q', B] \in \delta(q, a, A)$ *and you can pop* $A$ *and do push* $B$





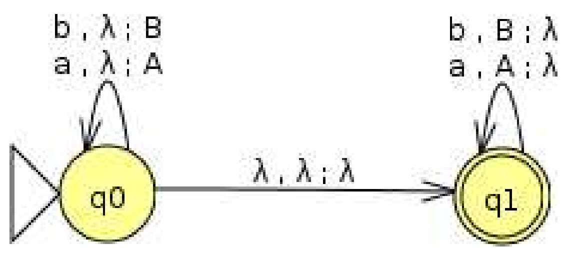

$$L(M) = \{w \in \Sigma^* \mid [q_0, w, \lambda] \vdash^* [q_i, \lambda, \lambda] \ \& \ q_i \in F\}$$


*Accepts* $\{a^n b^n \mid n \geq 0\}$

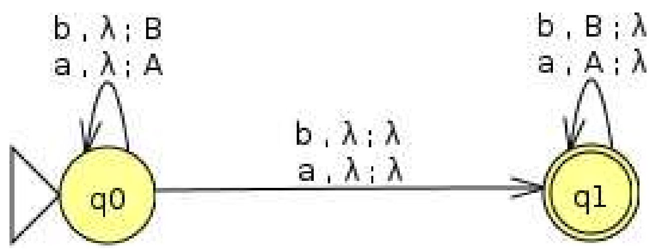



*accepts* $\{a^n \mid n \geq 0\} \cup \{a^n b^n \mid n \geq 0\}$

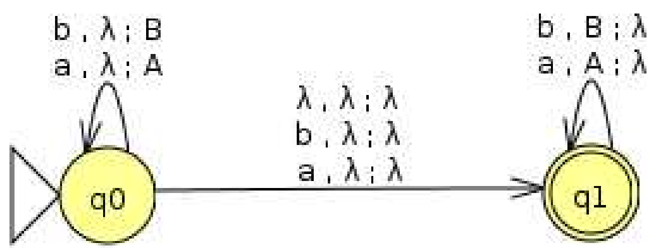
*Non-determinism is essential: in general it cannot be eliminated*




*accepts* $\{uu^R \in \Sigma^* \mid u \in \Sigma^*\}$ *the even palindromes*

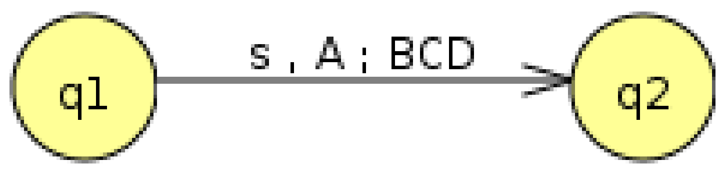



*accepts* $\{u\sigma u^R \in \Sigma^* \mid u \in \Sigma^*,\ \sigma \in \Sigma\}$ *the odd palindromes*




*accepts the palindromes*

*A PDA is* extended *if one may push a word over the stack alphabet*

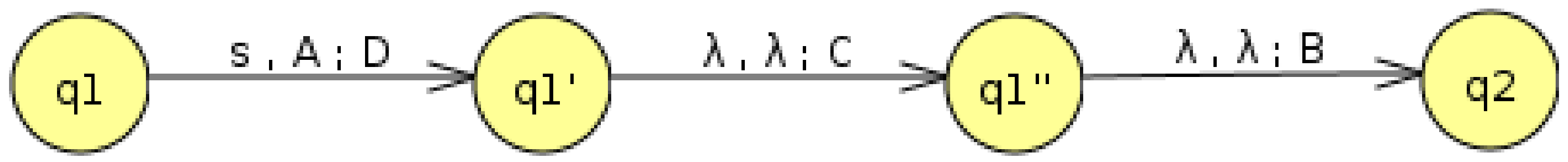



*formally* $[q_2, BCD] \in \delta(q_1, s, A)$*, and in general*

$$\delta : Q \times (\Sigma \cup \{\lambda\}) \times (\Gamma \cup \{\lambda\}) \to \mathcal{P}(Q \times \Gamma^*)$$

*Prop. Extended PDAs accept the same class of languages as PDAs*

*Proof. Replace a transition like above by*



*In a PDA we have by definition*

$$w \in L(M) \;\Leftrightarrow\; [q_0, w, \lambda] \vdash^* [q_i, \lambda, \lambda] \;\&\; q_i \in F$$

*That is* acceptance by empty stack and final state

*Alternatively we can define*

$$w \in L_F(M) \;\Leftrightarrow\; [q_0, w, \lambda] \vdash^* [q_i, \lambda, \alpha] \;\&\; q_i \in F$$

acceptance by final state

*or*

$$w \in L_E(M) \;\Leftrightarrow\; [q_0, w, \lambda] \vdash^+ [q_i, \lambda, \lambda]$$

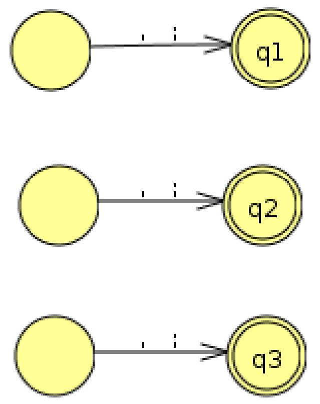
acceptance by empty stack

*Proposition* *Let $M$ be a PDA.*

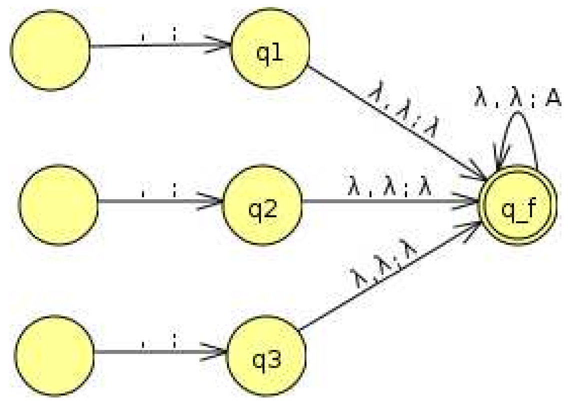
(i) *For each $L \in L_F(M)$ there is an $M'$ such that $L \in L(M')$.*

(ii) *For each $L \in L_E(M)$ there is an $M'$ such that $L \in L(M')$.*

*Proof sketch for (i):* *Transform an $M$ like*




*into*




*Proposition*

(i) *PDAs with acceptance by final state accept the same class of languages as PDAs*

(ii) *PDAs with acceptance by empty stack accept the same class of languages as PDAs*

*Theorem Let $L$ be a language over $\Sigma^*$.*

*Then the following are equivalent.*

(i) $L = L(G)$ *for some context-free grammar* $G$

(ii) $L = L(M)$ *for some PDA* $M$

*We will not prove this, but some intuition comes from exercises.* ■

*Let $L$ be a context-free language over $\Sigma^*$*

*Then there exists a number $k > 0$*

*such that every word $z \in L$ with $|z| > k$*

*can be written as $z = u_1 v_1 w v_2 u_2$ such that*

(i) $|v_1 w v_2| \leq k$
(ii) $|v_1| + |v_2| > 0$
(iii) $u_1 v_1^i w v_2^i u_2 \in L$ *for all* $i \geq 0$

*Proofsketch. If there is a sufficiently large word $z \in L$ then in the derivation of $z$ one has*

$$\begin{array}{lcl} S & \rightarrow & u_1 A u_2 \\ A & \rightarrow & v_1 A v_2 \\ A & \rightarrow & w \end{array}$$

*a grammar loop!*

*so that $S \Rightarrow u_1 v_1 w v_2 u_2 = z$ and also*

$$S \Rightarrow u_1 A u_2 \Rightarrow u_1 v_1 A v_2 u_2 \Rightarrow \cdots u_1 v_1^i A v_2^i u_2 \Rightarrow u_1 v_1^i w v_2^i u_2 \in L \blacksquare$$

- $L_1 := \{a^k b^k c^k \mid k \geq 0\}$ *is not context-free*

  *because it violates the pumping lemma for CF languages.*

- $L_2 := \{ww \mid w \in \{a,b\}^*\}$ *is not context-free*

  *because it violates the pumping lemma for CF languages.*

- *NB:* $L_3 := \{a^k b^k \mid k \geq 0\}$ *is context-free hence satisfies the pumping lemma for CF languages.*