4. Non-regular languages and the pumping lemma

Exercise 4.3 can be made and handed in, (*) is an extra exercise. Let \( \Sigma = \{a, b\} \).

4.1. 1. Construct a DFA that accepts \( L \) where \( L = L(b^*ab^*a^*b^*a^*b^*) \)
2. Derive (just like in the pumping lemma) from your automaton a number \( k \), for which you can prove:
   for every \( w \in L \) with \(|w| \geq k\), there are words \( u_1, v, u_2 \) such that
   - \( w = u_1vu_2 \)
   - \(|v| \geq 1\)
   - \(|u_1v| \leq k\)
   - \( \forall n \in \mathbb{N}\) (\( u_1v^n u_2 \in L \)).

Prove this.

4.2. 1. Prove that the language \( L_1 \) is not regular, where
   \[ L_1 := \{a^n b^p \mid p > n\} \]
2. Prove that the language \( L_2 \) is not regular, where
   \[ L_2 := \{a^p b^n \mid p > n\} \]

Try to give a proof without using the pumping lemma.

4.3. Now, \( \Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, +, \times, (, )\} \) Prove that the language \( L_3 \)
   is not regular, where
   \[ L_3 := \{e \in \Sigma^* \mid e \text{ is a well-formed arithmetical expression}\} \]

NB. In a well-formed arithmetical expression the brackets should “match”,
so \( 3 \times (5 + (3 + 0)) \) is well-formed and so is \( (((4 + 5) \times 7)), \) but \( 5 + 9 + 3 \)
and \( 4 \times (3 \times 7) \) are not.

4.4. (*) Do Exercise 5.4.2. from the Lecture Notes of Fiore (see the webpage of the course).