Homework Complexity Theory

February 18, 2020

To be handed in on February 28, 2020, in the delivery box of your TA in front of room M1.07A. Deadline: 11:00 AM.

Exercise 1. Given $T(n) = T(\lfloor n/3 \rfloor) + T(\lfloor n/4 \rfloor) + \Theta(n)$.

Prove that $T(n) = \mathcal{O}(n)$.

Solution. Let N_0 and C_0 be such that $\forall n > N_0.\ T(n) \leq T(\lfloor n/3 \rfloor) + 2T(\lfloor n/4 \rfloor) + C_0 n$. We need to find N_0 and C such that for all $n > N_0$. $T(n) \leq Cn$. Take $N := N_0, C := 6C_0$ with $n > N$. Then

$$\begin{aligned}
T(n) &\leq T(\left\lfloor \frac{n}{3} \right\rfloor) + 2T(\left\lfloor \frac{n}{4} \right\rfloor) + C_0 n \\
&\overset{IH}{\leq} C\frac{n}{3} + 2C\frac{n}{4} + C_0 n \\
&= \frac{C}{3}n + \frac{C}{2}n + C_0 n \\
&= \frac{5}{6}Cn + C_0 n \\
&= \frac{5}{6}Cn + \frac{1}{6}n \\
&= Cn.
\end{aligned}$$

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Exercise 2. Let $T(n) = 9T(n/2) + \Theta\left(n^3\right)$. Prove:

(a) $T(n) = \mathcal{O}\left(n^3\sqrt{n}\right)$, and

(b) $T(n) = \Omega\left(n^3 \log n\right)$.

Solution. (a) Let N_0, C_0 be such that for all $n > N_0$. $T(n) \leq 9T(n/2) + C_0 n^3$. Take $N := N_0$ and $C := 5C_0$ and let $n > N_0$. Then

$$\begin{aligned}
T(n) &\leq 9T(n/2) + C_0 n^3 \\
&\overset{IH}{\leq} 9C(n/2)^3\sqrt{n/2} + C_0 n^3 \\
&= \frac{9}{8\sqrt{2}}Cn^3\sqrt{n} + C_0 n^3 \\
&= \frac{9\sqrt{2}}{16}Cn^3\sqrt{n} + C_0 n^3 \\
&\leq \left(\frac{9\sqrt{2}}{16}C + C_0\right) n^3\sqrt{n} \\
&= \left(\frac{9\sqrt{2}}{16}C + \frac{1}{5}C\right) n^3\sqrt{n} \\
&\leq Cn^3\sqrt{n}.
\end{aligned}$$

To find the value of C we proceed by:

$$\frac{9\sqrt{2}}{16}C + C_0 \le C \iff C_0 \le C\left(1 - \frac{9\sqrt{2}}{16}\right)$$
$$\iff C \ge \frac{C_0}{1 - \frac{9\sqrt{2}}{16}}$$
we can take $C = 5C_0$.

(b) Let N_0, C_0 be such that for all $n > N_0$. $T(n) \ge 9T(n/2) + C_0 n^3$. Take $N := N_0, C := \frac{8}{9}C_0$ and let $n > N$ then

$$\begin{aligned}
T(n) &\ge 9T(n/2) + C_0 n^3 \\
&\overset{IH}{\ge} 9C(\frac{n}{2})^3 \log{(n/2)} + C_0 n^3 \\
&= \frac{9}{8}Cn^3(\log n - 1) + C_0 n^3 \\
&= \frac{9}{8}Cn^3 \log n + n^3\left(C_0 - \frac{9}{8}C\right) \\
&\ge Cn^3 \log n.
\end{aligned}$$

To find the value of C we proceed by:

$$C - \frac{9}{8}C \ge 0 \iff C_0 \ge \frac{9}{8}C$$
$$\iff C \le \frac{8}{9}C_0.$$

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Exercise 3. For $i = 2, 3, 4, 5$ the function T_i is given by $T_i(i) = i$ and $T_i(n) = 9T_i(n/i) + n^2$.
Determine functions g_i such that $T_i(n) = \Theta\left(g_i(n)\right)$ for $2 \le i \le 5$.

Solution. We apply the Master Theorem which distinguishes between the values of $\log_i 9$ for $i = 2, 3, 4, 5$. We need to check the conditions for the application of the Master Theorem in each one of these cases:

Case $i = 2$. We have $\log_2 9 > 3$, so $n^2 \in \mathcal{O}\left(n^{\log_2 9 - \varepsilon}\right)$ for some $\varepsilon > 0$. The existence of such ε can be checked by setting $\varepsilon = \log_2 9 - 2 \approx 1.169\ldots$. Therefore, by the Master Theorem (Case 1), $T_2(n) = \Theta\left(n^{\log_2 9}\right)$ and then $g_2(n) = n^{\log_2 9}$.

Case $i = 3$. We have $\log_3 9 = 2$, so $n^2 \in \Theta\left(n^{\log_3 9}\right)$ trivially. Therefore, by the Master Theorem (Case 2), $T_3(n) = \Theta\left(n^2 \log n\right)$ and $g_3(n) = n^{\log_3 9}$.

Case $i = 4$. We have $\log_4 9 \approx 1.584\ldots$, so $n^2 \in \mathcal{O}\left(n^{\log_4 9 + \varepsilon}\right)$ for some $\varepsilon > 0$. Therefore, by the Master Theorem (Case 3), $T_4(n) = \Theta\left(n^2\right)$. Notice that we still need to check one more condition for case 3, i.e., there exists $C < 1$ such that $f(n/4) \le Cf(n)$. This is basically $\frac{9}{16}n^2 \le Cn^2$, just take $C = \frac{9}{16}$. So, we have that $g_4(n) = n^{\log_4 n}$.

Case $i = 5$. We have $\log_5 9 \approx 1.365\ldots$, so $n^2 \in \mathcal{O}\left(n^{\log_5 9 + \varepsilon}\right)$ for some $\varepsilon > 0$. Therefore, by the Master Theorem (Case 3), $T_5(n) = \Theta\left(n^2\right)$. We need to check the conditions of case 3, i.e., there exists $C < 1$ such that $9f(n/5) \le Cf(n)$. This is basically $\frac{9}{25}n^2 \le Cn^2$, just take $C = \frac{9}{25}$, so $g_5(n) = n^{\log_5 9}$.

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Exercise 4. Let $T(n) = T(n/2) + 2T(n/4) + 6n$.
Prove that $T(n) = \mathcal{O}(n \log n)$.

Solution. We need to find a constant C and a natural number N_0 such that for all $n > N_0$, $T(n) \leq Cn \log n$. Let us take $C \geq 4, N = N_0$. We proceed the reasoning by induction as follows:

$$\begin{aligned}
T(n) &= T(n/2) + 2T(n/4) + 6n \\
&\overset{IH}{\leq} C\frac{n}{2}\log(n/2) + \frac{C}{2}n\log(n/4) + 6n \\
&= \frac{C}{2}n(\log n - 1) + \frac{C}{2}n(\log n - 2) + 6n \\
&= \frac{C}{2}n\log n - \frac{C}{2}n + \frac{C}{2}n\log n - Cn + 6n \\
&= Cn\log n - \frac{C}{2}n - Cn + 6n \\
&= Cn\log n - \frac{3}{2}Cn + 6n \\
&= Cn\log n + \left(6 - \frac{3}{2}C\right)n \\
&\leq Cn\log n, \text{ for } C \geq 4.
\end{aligned}$$

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Exercise 5. Let $T(n) = 2T(n-1) + 2T(n-2) + 3T(n-3) + \mathcal{O}(n)$.
Prove that $T(n) = \mathcal{O}(3^n)$.

Solution. Let C_0, N_0 be such that $\forall n > N_0$. $T(n) \leq 2T(n-1) + 2T(n-2) + 3T(n-2) + C_0 n$. We want to prove $T(n) \leq C3^n$ for some C and n sufficiently large. A direct proof by induction does not work, so we prove $T(n) \leq C3^n - dn$ by induction.

$$\begin{aligned}
T(n) &\leq 2T(n-1) + 2T(n-2) + 3T(n-3) + C_0 n \\
&\overset{IH}{\leq} 2(C3^{n-1} - d(n-1)) + 2(C3^{n-2} - d(n-2)) + 3(Cn3^{n-3} - d(n-3)) + C_0 n \\
&= \frac{2}{3}C3^n - 2d(n-1) + \frac{2}{9}C3^n - 2d(n-2) + \frac{3}{27}C3^n - 3(d(n-3)) + C_0 n \\
&= C3^n - 7dn + 15d + C_0 n, \text{ take } d = C_0 \\
&= C3^n - 7C_0 n + 15C_0 + C_0 n \\
&= C3^n + C_0(15 - 6n) \\
&\leq C3^n, \text{ for } n \geq 3 \text{ and } C > 1.
\end{aligned}$$

So, with $C > 1$ and $d = C_0$, we have proved $T(n) \leq C3^n - dn$. Therefore, $T(n) = \mathcal{O}(3^n)$ for all $n \geq 3$. ■