Exercises Complexity Theory

Lecture 1

April 11, 2023

Only the exercises where points are given can be handed in.
(The maximum number of points per exercise is written in the margin.)

To be handed in on **April 17, 2023**, in Brightspace under Assignment 1, **deadline: 10:00 AM.**

Exercise 1. Let $f = \mathsf{fib}$ be defined by $f(i) = i$ for $i = 0, 1$ and $f(i) = f(i-1) + f(i-2)$ for $i > 1$.

(1) (a) Prove by induction on n that

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} f(n-1) & f(n) \\ f(n) & f(n+1) \end{pmatrix}$$

for all $n \geq 1$.

(b) Prove by induction on n that

$$\begin{aligned} f(n-1)f(n+1) &= f(n)^2 + 1, \text{ if } n \text{ is even} \\ f(n-1)f(n+1) &= f(n)^2 - 1, \text{ if } n \text{ is odd.} \end{aligned}$$

Exercise 2. Let g be defined by $g(i) = 1$ for $i = 0, 1, 2$ and $g(i) = g(i-2) + g(i-3)$ for $i > 2$. Prove by induction on n that $g(n) > 0.5 \times (1.2)^n$ for all $n \geq 0$.

Exercise 3. Given $T(n) = 2T(\lfloor n/3 \rfloor) + 2T(\lfloor n/6 \rfloor) + d\,n$, where $d \geq n$. (So d is some fixed constant.)

(2) (a) Prove that $T(n) = \mathcal{O}(n \log n)$, where you take rounding off errors into account.

(b) Prove that $T(n) = \Omega(n \log n)$, where you take rounding off errors into account.

Exercise 4. Let $T(n) = 10T(n/3) + \Theta\left(n^2\right)$. Prove the following using the Substitution Method (where you may ignore rounding off errors).

(2) (a) $T(n) = \mathcal{O}\left(n^2\sqrt{n}\right)$, and

(2) (b) $T(n) = \Omega\left(n^2 \log_3 n\right)$.

(3) **Exercise** 5. Let $T(n) = 7T(n-2) + 6T(n-3) + \Theta\left(n^2\right)$.
Prove that $T(n) = \Theta(3^n)$.

Exercise 6. Rank the following functions in n by order of growth from low to high; some may be of the same order.

$$n\sqrt{n} \qquad \sum_{i=0}^{n} \log n \qquad n^n \qquad \log\sqrt{n} \qquad \log(n^2) \qquad (\log n)^2 \qquad 2^n \qquad 3^n$$

$$\sum_{i=0}^{\lfloor \log n \rfloor} i \qquad \sum_{i=0}^{n} i^2 \qquad n^{0.001} \qquad 17n^3 \qquad 17^{\log 89} \qquad n^2 \qquad 100n \qquad 1$$

Exercise 7. For each of the following functions, $p(n)$, give the function $q(n)$ from the list such that $p(n) = \Theta(q(n))$.

$$f(n) = \sum_{i=1}^{n}(4i-4) \qquad g(n) = \sum_{i=1}^{n}\sum_{j=1}^{i} i \qquad h(n) = \sum_{i=1}^{\lfloor \log n \rfloor} n \qquad k(n) = \sum_{i=0}^{n} \frac{4}{2^i}$$

$$q(n) = \quad 1 \qquad n \qquad n(\log n)^2 \qquad n^2 \log n \qquad 2^n \qquad n^n \qquad \log n \qquad n \log n \qquad n^2 \qquad n^3$$