Exercises Complexity Theory

Lecture 5

May 15, 2023

Only the exercises where points are given can be handed in.
(The maximum number of points per exercise is written in the margin.)

To be handed in on **May 22, 2023**, in Brightspace under Assignment 5, **deadline: 10:00 AM.**

Exercise 1. A CNF φ is called *pre-satisfiable* if there is an assignment $v : \text{Atoms}(\varphi) \to \{0, 1\}$ for which all clauses in φ are true, except for at most one.

CNF-preSAT is the problem to decide for a CNF whether it is pre-satisfiable.

In this exercise, we prove that CNF-preSAT is NP-complete.

(5) (a) Describe precisely what has to be done to prove that CNF-preSAT is NP-complete.

(20) (b) Prove that CNF-preSAT is NP-complete.

Exercise 2. For $C \subseteq \mathbb{Z}$, we consider $\mathsf{ILP}(C)$ which is a variant of the *integer linear programming problem*, ILP, that we have seen in the course. Given a finite set E of inequalities of the form

$$a_1x_1 + \ldots + a_nx_n \leq c$$

where $c \in C$ and $a_1, \ldots, a_n \in \mathbb{Z}$, we ask if there are values $x_1, \ldots, x_n \in \mathbb{Z}$ such that all inequalities in E hold. In the course, we have seen that $\mathsf{ILP}(\mathbb{Z})$ is NP-complete.

(10) (a) Give a set $C \subseteq \mathbb{Z}$, with C containing at least two elements, for which $\mathsf{ILP}(C)$ is not NP-complete.

(25) (b) Show that for $C = \{-1, 1\}$, $\mathsf{ILP}(C)$ is NP-complete.
Hint Add a fresh variable x to each equation to transform an inequality $a_1x_1 + \ldots + a_nx_n \leq b$ to $a_1x_1 + \ldots + a_nx_n + ax \leq 1$; add additional equalities for x that force x to be 1.

Exercise 3. Define, for $G = (V, E)$ an undirected graph, the problem "relaxed 3Color", $r\mathsf{3Color}(G)$, as the problem to decide whether G can be 3-colored where **at most one edge can have both endpoints of the same color** and all other edges have two endpoints with a different color.

(5) (a) Give a graph G such that $r\mathsf{3Color}(G)$ holds, but $\mathsf{3Color}(G)$ does not hold.

(10) (b) Define precisely, in a formula, when $r\mathsf{3Color}(G)$ holds, and state precisely what we need to prove to show that r3Color is NP-complete.

(25) (c) Prove that r3Color is NP-complete.
Hint Use the NP-completeness of 3Color; add a simple graph to your graph.

Exercise 4. For $G = (V, E)$ a graph, an **independent set** of G is a set X of vertices such that there is no edge from x to y if both x and y are members of X.

(a) Give an example of a graph and two nonempty sets of vertices of which one is an independent set and the other is not.

(b) Show that the following decision problem is **NP**-complete:
Given a graph G and an integer k, is there an independent set with k vertices?
Hint Consider the problem $\mathsf{Clique}(k)$.