Exercises Complexity Theory

Lecture 6

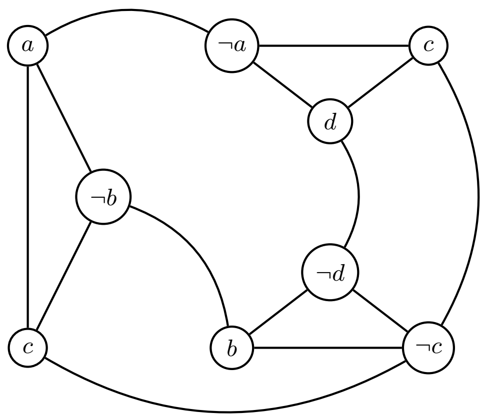
May 22, 2023

Only the exercises where points are given can be handed in.
(The maximum number of points per exercise is written in the margin.)

To be handed in on **June 5, 2023**, in Brightspace under Assignment 6, **deadline: 10:00 AM.**

Exercise 1. Given an undirected graph $G = (V, E)$, the set of vertices $V' \subseteq V$ is called **independent** if no two vertices of V' are connected by an edge. $\mathsf{Indep}(G, k)$ is the problem to decide whether for graph G and number k, the graph G contains an independent set of k vertices. We show that Indep is **NP**-complete using 3CNF-SAT.

(6) (a) In the reduction from 3CNF-SAT to Indep, we define, for a 3CNF formula $\varphi = \bigwedge_{i=1}^{k} C_i$, a graph with $3k$ vertices. The construction is illustrated by the following example: If $\varphi = (a \vee \neg b \vee c) \wedge (\neg a \vee c \vee d) \wedge (b \vee \neg c \vee \neg d)$, then the associated graph is:



Give three satisfying assignments $v : \{a, b, c, d\} \to \{0, 1\}$ for φ and for each of these a "corresponding" independent set of size 3.

(5) (b) Use the construction in part (a) to define a reducing map f from CNFs to graphs. Define it precisely.

(12) (c) Use the map f from part (b) to prove **NP**-hardness of Indep using the **NP**-hardness of 3CNF-SAT.

Exercise 2. Say for each of the following problems whether they are in **P** or **NP**-hard. Prove your answer.

(12) (a) Given an undirected graph $G = (V, E)$, do we have $\mathsf{2Color}(G)$? (NB. $\mathsf{2Color}(G)$ states that G is 2-colorable.)

(12) (b) Given an undirected graph $G = (V, E)$, do we have $\mathsf{4Color}(G)$? (NB. $\mathsf{4Color}(G)$ states that G is 4-colorable.)

(12) (c) Given an undirected graph $G = (V, E)$, do we have $\mathsf{Clique\text{-}2Cover}(G)$? (NB. $\mathsf{Clique\text{-}2Cover}(G)$ states that V is the union of 2 cliques: $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ and each V_i is a clique.)

(12) (d) Given an undirected graph $G = (V, E)$, do we have $\mathsf{Clique\text{-}4Cover}(G)$? (NB. $\mathsf{Clique\text{-}4Cover}(G)$ states that V is the union of 4 cliques: $V = V_1 \cup V_2 \cup V_3 \cup V_4$ with all V_i disjoint and each V_i is a clique.)

Exercise 3. A graph (V_1, E_1) is a *subgraph* of a graph (V_2, E_2) if there is an injective mapping $h : V_1 \to V_2$ such that

$$\forall (u, v) \in E_1 \, (h(u), h(v)) \in E_2.$$

The decision problem $\mathsf{Subgraph}(G_1, G_2)$ is: given two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, is G_1 a subgraph of G_2?

(a) Give an example of two graphs G_1 and G_2 for which $\mathsf{Subgraph}(G_1, G_2)$ holds and an example of two graphs G_1 and G_2 for which $\mathsf{Subgraph}(G_1, G_2)$ does not hold.

(12) (b) Prove that Subgraph is **NP**-complete, using the fact that Clique is **NP**-complete. (First describe precisely what has to be proven.)

Exercise 4. For a graph $G = (V, E)$, a *nearly-Hamilton cycle* is a cycle in the graph G that has exactly one vertex occurring twice, and all others occurring exactly once. The problem $\mathsf{nearHam}(G)$ is to decide whether G has a nearly-Hamilton cycle.

(5) (a) Give a graph $G = (V, E)$ that has a nearly-Hamilton cycle, but not a Hamilton cycle.

(12) (b) Prove that nearHam is **NP**-complete.