Exercises Complexity Theory

Lecture 1

April 7, 2025

Only the exercises where points are given can be handed in.
(The maximum number of points per exercise is written in the margin.)

To be handed in on **April 14, 2025**, in Brightspace under Assignment 1, **deadline: 11:00 AM.**

Exercise 1. We prove a number of standard useful properties about log. Suggestion: **remember these**. First of all, here are some facts from which basically everything follows.

- By definition:
$$\log_b a = x \quad \Longleftrightarrow \quad b^x = a$$
- Hence, the exponent and logarithm functions are inverses:
$$b^{\log_b a} = a \quad \text{and} \quad \log_b b^a = a.$$
- Hence, the exponent and logarithm functions are injective:
$$b^x = b^y \implies x = y \quad \text{and} \quad \log_b x = \log_b y \implies x = y.$$
- The exponent and logarithm functions are strictly order-preserving:
$$x < y \implies b^x < b^y \quad \text{and} \quad x < y \implies \log_b x < \log_b y.$$

Let $a, b, c > 0$.

(a) Prove the following:
$$\log_a b \cdot \log_b c = \log_a c.$$
(Hint: consider a^x for x the two sides of the equation.)

(b) From (a), show that one can "change the logarithm base at a constant cost factor", by showing that
$$\log_b c = d \log_a c \text{ for some constant } d > 0.$$

(c) Prove the following:
$$\log_b a^c = c \log_b a.$$

(d) From (a) and (c), show that
$$a^{\log_c b} = b^{\log_c a}.$$
(Hint: take $\log_a$ at both sides of this equation.)

(e) Suppose $a > 1$. Given $d > 0$, prove that
$$\log_a x + d > \log_a (x + d)$$
for sufficiently large x. When is x "sufficiently large"? (Note that this also implies $\log_a(x - d) > \log_a x - d$. These inequalities are sometimes helpful in approximations to commute log with +.)

Exercise 2. Rank the following functions in n by order of growth from low to high; some may be of the same order. We view f as being "lower than" g in case $f = \mathcal{O}(g)$ and $g \neq \mathcal{O}(f)$.

$$n\sqrt{n} \qquad \sum_{i=0}^{n} \log n \qquad n^n \qquad \log\sqrt{n} \qquad \log(n^2) \qquad (\log n)^2 \qquad 2^n \qquad 3^n$$

$$\sum_{i=0}^{\lfloor \log n \rfloor} i \qquad \sum_{i=0}^{n} i^2 \qquad n^{0.001} \qquad 17n^3 \qquad 17^{\log 89} \qquad n^2 \qquad 100n \qquad 1$$

Exercise 3. Prove that every $n \geq 1$ is either a power of 2 or can be written as the sum of powers of 2 using strong induction.

Exercise 4. Let g be defined by $g(0) = 0$, $g(1) = 5$, $g(2) = 3$, and $g(n+3) = 2g(n+2) + g(n+1) - 2g(n)$ for $n \geq 3$.

(10) (a) Use the method of the lecture to derive a closed expression for $g(n)$ (that is: express $g(n)$ in terms of n, without recursion).

(10) (b) Show that g is exponential by giving a such that $g(n) = \Theta(a^n)$.

Exercise 5. Let $T(n) = 4T(\lfloor n/6 \rfloor) + T(\lfloor n/3 \rfloor) + d\,n$, where $d \geq 0$ d is some fixed constant.

(20) (a) Prove that $T(n) = \mathcal{O}(n \log n)$, where you take rounding off errors into account.

(b) Prove that $T(n) = \Omega(n \log n)$, where you take rounding off errors into account.

Exercise 6. Let $T(n) = 10T(\frac{n}{3}) + \Theta\left(n^2\right)$. Prove the following using the Substitution Method (where you may ignore rounding off errors).

(a) $T(n) = \mathcal{O}\left(n^2\sqrt{n}\right)$, and

(b) $T(n) = \Omega\left(n^2 \log_3 n\right)$.

Exercise 7. Let $T(n) = 3T(n-1) + 4T(n-2) + \Theta\left(n^3\right)$. Prove the following using the Substitution Method (where you may ignore rounding off errors).

(30) (a) $T(n) = \mathcal{O}(4^n)$. (Hint: you may need to add an additional term $-dn^3$.)

(30) (b) $T(n) = \Omega(4^n)$.