Exercises Complexity Theory

Lecture 5

May 19, 2025

Only the exercises where points are given can be handed in.
(The maximum number of points per exercise is written in the margin.)

To be handed in on **May 26, 2025**, in Brightspace under Assignment 5, **deadline: 11:00.**

Exercise 1. For $k \in \mathbb{N}$, a CNF φ is called *k-pre-satisfiable* if there is an assignment $v:\ \text{Atoms}(\varphi) \to \{0,1\}$ for which all clauses in φ are true, except for at most k.

CNF-preSAT(k) is the problem to decide for a CNF whether it is k-pre-satisfiable.

In this exercise, we prove that CNF-preSAT(k) is NP-complete for all $k \in \mathbb{N}$.

(5) (a) Describe precisely what has to be done to prove that CNF-preSAT(k) is NP-complete.

(20) (b) Prove that CNF-preSAT(k) is NP-complete.

Exercise 2. For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we say that G_1 and G_2 are *isomorphic* if there is a bijective function $f\colon V_1 \to V_2$ such that $(u, v) \in E_1 \iff (f(u), f(v)) \in E_2$.

Further, a *subgraph* of a graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$, and $E' \subseteq E$.

(10) (a) Give an example of a graph $G = (V, E)$ with at least 4 vertices, and a subgraph $G' = (V', E')$ of G with $|V'| = |V| - 1$. Also, give a graph which is not isomorphic to any subgraph of G.

(25) (b) Show that the following decision problem is **NP**-complete:

Given graphs G_1, G_2, is G_1 isomorphic to some subgraph of G_2?

Hint: consider the problem Clique(k).

Exercise 3. Recall the following decision problem (Halt):

Given a Turing machine T and input x, does T halt on x?

(10) (a) Show that Halt is **NP**-hard.

(5) (b) Is Halt **NP**-complete?

Exercise 4. Let $G = (V, E)$ be a graph. An n**-coloring** of G is is a function $c\colon V \to \{1, \ldots, n\}$ such that $c(u) \neq c(v)$ for all $(u, v) \in E$. Let nColor be the following decision problem:

Given a graph $G = (V, E)$, does G have an n-coloring?

(5) (a) Show that for any $n \in \mathbb{N}$, the problem nColor is in **NP**.

(20) (b) Show that for any $n \geq 3$, the problem nColor is **NP**-complete.

Hint: Use induction and show that for any $n \in \mathbb{N}$, we have $n\text{Color} \leq_P (n+1)\text{Color}$.

Exercise 5. For a graph $G = (V, E)$, an **r-independent set** of G is a set X of vertices such that there is exactly one pair of vertices $u, v \in X$ with an edge between them.

(a) Give an example of a graph and two nonempty sets of vertices of which one is an r-independent set and the other is not.

(b) Show that the following decision problem is **NP**-complete:
Given a graph G and an integer $k > 2$, is there an r-independent set with k vertices?
Hint: Consider (a slight modification of) the problem Clique(k).