Exercises Complexity Theory

Lecture 6

May 26, 2025

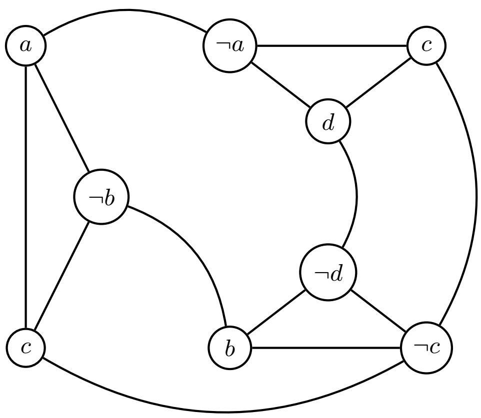
Only the exercises where points are given can be handed in.
(The maximum number of points per exercise is written in the margin.)

To be handed in on **June 2, 2025**, in Brightspace under Assignment 6, **deadline: 11:01.**

Exercise 1. Given an undirected graph $G = (V, E)$, the set of vertices $V' \subseteq V$ is called **independent** if no two vertices of V' are connected by an edge. $\mathsf{Indep}(G, k)$ is the problem to decide whether for graph G and number k, the graph G contains an independent set of k vertices. We show that Indep is **NP**-complete in two ways: by a reduction from 3CNF-SAT and from Clique.

(2) (a) Show that Indep is in **NP**.

(6) (b) In the reduction from 3CNF-SAT to Indep, we define, for a 3CNF formula $\varphi = \bigwedge_{i=1}^{k} C_i$, a graph with $3k$ vertices. The construction is illustrated by the following example: If $\varphi = (a \vee \neg b \vee c) \wedge (\neg a \vee c \vee d) \wedge (b \vee \neg c \vee \neg d)$, then the associated graph is:



Give three satisfying assignments $v : \{a, b, c, d\} \to \{0, 1\}$ for φ and for each of these a "corresponding" independent set of size 3.

(5) (c) Use the construction in part (a) to define a reducing map f from CNFs to graphs. Define it precisely.

(12) (d) Use the map f from part (b) to prove **NP**-hardness of Indep using the **NP**-hardness of 3CNF-SAT.

(5) (e) Prove that Indep is **NP**-hard using a reduction from Clique.

Exercise 2. Say for each of the following problems whether they are in **P** or **NP**-hard. Give a proof sketch of your answer.

(10) (a) Given a set of elements $X = \{x_1, \ldots, x_n\}$, each with a weight w_i and a value v_i and a knapsack with a carry capacity of C, can we find a subset $X' \subseteq X$ such that

$$\sum_{x_i \in X'} w_i \leq C \text{ and } \sum_{x_i \in X'} v_i \geq V$$

for some target value V?

(10) (b) Given a graph $G = (V, E)$, does the graph consist of k connected components?

(10) (c) Given a connected graph $G = (V, E)$, does G have a 2-coloring (i.e. a function $c\colon V \to \{0, 1\}$ such that for all $(u, v) \in E$, we have $c(u) \neq c(v)$)?

(10) (d) Given a graph $G = (V, E)$ and a natural number $n \geq 3$, is G the union of n cliques?

Exercise 3. For a graph $G = (V, E)$, a *nearly-Hamilton cycle* is a cycle in the graph G that has exactly one vertex occurring twice, and all others occurring exactly once. The problem $\mathsf{NearHam}(G)$ is to decide whether G has a nearly-Hamilton cycle.

(3) (a) Give a graph $G = (V, E)$ that has a nearly-Hamilton cycle, but not a Hamilton cycle.

(12) (b) Prove that $\mathsf{NearHam}$ is **NP**-complete.

Exercise 4. For $C \subseteq \mathbb{Z}$, we consider $\mathsf{ILP}(C)$ which is a variant of the *integer linear programming problem*, ILP, that we have seen in the course. Given a finite set E of inequalities of the form

$$a_1 x_1 + \ldots + a_n x_n \leq c$$

where $c \in C$ and $a_1, \ldots, a_n \in \mathbb{Z}$, we ask if there are values $x_1, \ldots, x_n \in \mathbb{Z}$ such that all inequalities in E hold. In the course, we have seen that $\mathsf{ILP}(\mathbb{Z})$ is NP-complete.

(3) (a) Show that $\mathsf{ILP}(\{0\})$ can be solved in constant time.

(12) (b) Show that $\mathsf{ILP}(\{-1, 1\})$ is NP-complete.