Exercises Complexity Theory

Lecture 5

May 11, 2026

Only the exercises where points are given can be handed in.
(The maximum number of points per exercise is written in the margin.)

To be handed in on **May 26, 2026**, in Brightspace under Assignment 5, **deadline: 11:00.**

Exercise 1. For $k \in \mathbb{N}$, a CNF φ is called *k-pre-satisfiable* if there is an assignment $v : \text{Atoms}(\varphi) \to \{0, 1\}$ for which all clauses in φ are true, except for at most k.

CNF-preSAT(k) is the problem of deciding for a CNF whether it is k-pre-satisfiable.

In this exercise, we prove that CNF-preSAT(k) is NP-complete for all $k \in \mathbb{N}$.

(5) (a) Describe precisely what has to be done to prove that CNF-preSAT(k) is NP-complete.

(15) (b) Prove that CNF-preSAT(k) is **NP**-complete.

Exercise 2. For $C \subseteq \mathbb{Z}$, we consider ILP(C) which is a variant of the *integer linear programming problem*, ILP, that we have seen in the course. Given a finite set E of inequalities of the form

$$a_1x_1 + \ldots + a_nx_n \leq c$$

where $c \in C$ and $a_1, \ldots, a_n \in \mathbb{Z}$, we ask if there are values $x_1, \ldots, x_n \in \mathbb{Z}$ such that all inequalities in E hold. In the course, we have seen that ILP($\mathbb{Z}$) is NP-complete.

(10) (a) Give a set $C \subseteq \mathbb{Z}$, with C containing at least two elements, for which ILP(C) is not NP-complete.

(15) (b) Show that for $C = \{-1, 1\}$, ILP(C) is NP-complete.
<u>Hint:</u> Add a fresh variable x to each equation to transform an inequality $a_1x_1 + \ldots + a_nx_n \leq b$ to $a_1x_1 + \ldots + a_nx_n + ax \leq 1$; add additional equalities for x that force x to be 1.

Exercise 3. Recall the following decision problem VertexCover(G, k): Given an undirected graph $G = (V, E)$ and a number k, is there a vertex cover of size k in G? A vertex cover is a set of points $W \subseteq V$ such that each edge has an endpoint (or both) in W.

(5) (a) A *full vertex cover* is a set of vertices F such that each edge has both endpoints in F. Show that deciding whether a graph has a full vertex cover of size k is in **P**.

(15) (b) A *relaxed vertex cover* is a set of vertices R such that every edge except for one has an endpoint (or both) in R. Show that deciding whether a graph has a relaxed vertex cover of size k is **NP**-complete.

Exercise 4. Let $G = (V, E)$ be a graph. An n**-coloring** of G is is a function $c \colon V \to \{1, \ldots, n\}$ such that $c(u) \neq c(v)$ for all $(u, v) \in E$. Let nColor be the following decision problem:

Given a graph $G = (V, E)$, does G have an n-coloring?

(5) (a) Show that for any $n \in \mathbb{N}$, the problem nColor is in **NP**.

(15) (b) Show that for any $n \geq 3$, the problem nColor is **NP**-complete.
<u>Hint:</u> Use induction and show that for any $n \in \mathbb{N}$, we have $n\text{Color} \leq_P (n+1)\text{Color}$.

Exercise 5. Recall the following decision problem (Halt):
Given a Turing machine T and input x, does T halt on x?

(10) (a) Show that Halt is **NP**-hard.

(5) (b) Is Halt **NP**-complete?