Complexity Resit Exam

July 13, 2022

This exam consists of four problems. Put your answers on the lined paper. Your solutions are judged not only on correctness, but also clarity. Good luck!

Problem 1 (20 points)

1. Two students are quibbling over the complexity of integer multiplication: the one says it's $\mathcal{O}(n^2)$, while the other claims it's $\mathcal{O}(n^{\log_2(3)})$. Who is right?

2. Suppose you are consulted by a company that wishes to improve an algorithm operating on large datasets that currently has running time

$$T(n) \;=\; 2\,T(\lceil 5/8 \cdot n \rceil) \;+\; 256n. \tag{1}$$

   After some consideration, you find two mutually exclusive methods, "A" and "B", to improve the algorithm's runtime:

   A. reduces the factor 256 in (1) to 32.

   B. reduces the factor $5/8$ in (1) to $3/8$.

   Which method yields the best asymptotic runtime? Which method will you pursue?

3. Is the variation on SAT, where the formulae are built from variables using the logical connectives $\wedge$ and $\vee$ (but not $\neg$) **NP**-complete? Briefly explain why.

4. Many problems can be encoded as integer linear programs, such as finding the maximum flow in a flow network. Someone claims that this means that we have a reduction from integer linear programming to the max-flow problem, and that therefore the max-flow problem is NP-complete. Are they correct?

Problem 2 (30 points) Write down a concrete asymptotic solution — for example, $T(n) = \Theta(n \lg(n))$ — for each of the following recurrence relations.

1. $T(n) \;=\; 25\,T(\lceil n/5 \rceil) + n^3$ for $n \geq 2$

2. $T(n) \;=\; 125\,T(\lceil n/5 \rceil) + n^3$ for $n \geq 2$

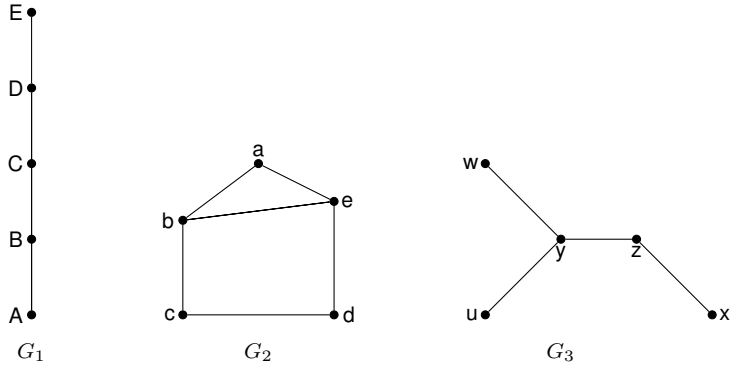
3. $T(n) \;=\; 625\,T(\lceil n/5 \rceil) + n^3$ for $n \geq 2$

4. $T(n) = 9\,T(\lceil n/3 \rceil + 2) + n$ for $n \geq 5$
5. $T(n) = T(\lceil n/2 \rceil) + 3\,T(\lfloor n/7 \rfloor) + n$ for $n \geq 2$
6. $T(n) = \sum_{k=1}^{\infty} T(\lfloor n/3^k \rfloor)$ for $n \geq 3$, and $T(0) = 0$

(You may assume that $T(n) > 0$ for all $n > 0$.)

Problem 3 (25 points) A **subgraph** of G is a graph G' whose vertices and edges are subsets of the vertices and edges of G respectively. We say that two graphs G_1 and G_2 are **isomorphic** if there is a bijective function $f : G_1 \to G_2$ such that we have an edge from x to y if and only if we have an edge from $f(x)$ to $f(y)$.

1. Is G_1 isomorphic to a subgraph of G_2? And of G_3?



2. Show that the following problem, known as Subgraph, is **NP**-complete:

   Given finite graphs G_1 and G_2, is G_1 isomorphic to a subgraph of G_2?

Problem 4 (25 points) In this problem, we look at coloring problems. More specifically, we define the problem kColor for every natural number k to be:

Given a finite graph G, does G have a k-coloring?

If $k = 3$, then this is the problem 3Color which we discussed during the lectures. (Note: in this exercise you may use the fact that 3Color is **NP**-complete.)

1. Is the problem 2Color in **P**? Briefly explain why.
2. Give a graph with 5 nodes and give a 4-coloring of that graph.
3. Show that for every k with $k \geq 3$ the problem kColor is **NP**-complete.