Matrix Calculations: Vector Spaces and Linear Maps

H. Geuvers

Institute for Computing and Information Sciences – Intelligent Systems
Radboud University Nijmegen

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Outline

Vector spaces

Linear independence & bases

Linear maps

Linear maps and matrices
Points in plane

- The set of points in a plane is usually written as
  \[ \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\} \]
  or as \( \mathbb{R}^2 = \{(\frac{x}{y}) \mid x, y \in \mathbb{R}\} \)
- Two points can be added, as in:
  \[(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)\]
  What is this geometrically?
- Also, points can be multiplied by a number (‘scalar’):
  \[a \cdot (x, y) = (a \cdot x, a \cdot y)\]
- Several nice properties hold, like:
  \[a \cdot ((x_1, y_1) + (x_2, y_2)) = a \cdot (x_1, y_1) + a \cdot (x_2, y_2)\]
Points in space

• Points in 3-dimensional space are described as:

\[ \mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\} \quad \text{or as} \quad \mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\} \]

• Again such 3-dimensional points can be added and multiplied:

\[
(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)
\]

\[
a \cdot (x, y, z) = (a \cdot x, a \cdot y, a \cdot z)
\]

And similar nice properties hold.

• We like to capture such similarities in a general abstract definition
  
  • sometimes the definition is so abstract one gets lost
  • but then it is good to keep the main examples in mind.
Vector space

Definition

A vector space consists of a set $V$, whose elements

- are called vectors
- can be added
- can be multiplied with a real number

The precise requirements are in later slides.

Example

For each $n \in \mathbb{N}$, $n$-dimensional space $\mathbb{R}^n$ is a vector space, where

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R}\}.$$ 

This includes the 2-dimensional plane ($n = 2$) and 3-dimensional space ($n = 3$).
Example

The set of solutions of a homogeneous system of equations is a vector space.

Solutions of a homogeneous system of equations

- can be added
- can be multiplied with a real number

to form new solutions.

(This is what we have seen last week.)

- Vector spaces occur at many places in many disguises.
- In general a vector space is a set $V$ with two operations “addition” and “scalar multiplication” that satisfy certain requirements.
Addition for vectors: precise requirements

1. Vector addition is **commutative**: summands can be swapped:
   \[ v + w = w + v \]

2. Addition is **associative**: grouping of summands is irrelevant:
   \[ u + (v + w) = (u + v) + w \]

3. There is a **zero vector** \( 0 \) such that:
   \[ v + 0 = v, \quad \text{and hence by 1 also:} \quad 0 + v = v. \]

4. Each vector \( v \) has an **additive inverse** (minus) \( -v \) such that:
   \[ v + (-v) = 0 \]
   One writes \( v - w \) for \( v + (-w) \).
Scalar multiplication for vectors: precise requirements

1  $1 \in \mathbb{R}$ is unit for scalar multiplication:

$$1 \cdot v = v$$

2  two scalar multiplications can be done as one:

$$a \cdot (b \cdot v) = (ab) \cdot v$$

twice scalar mult.  mult. in $\mathbb{R}$

3  distributivity

$$a \cdot (v + w) = (a \cdot v) + (a \cdot w)$$
$$(a + b) \cdot v = (a \cdot v) + (b \cdot v).$$

Exercise

Check for yourself that all these properties hold for $\mathbb{R}^n$. 
• In $\mathbb{R}^3$ we can distinguish three special vectors:

$$(1, 0, 0) \in \mathbb{R}^3 \quad (0, 1, 0) \in \mathbb{R}^3 \quad (0, 0, 1) \in \mathbb{R}^3$$

• These vectors form a basis: each vector $(x, y, z)$ can be expressed in terms of these three special vectors:

$$(x, y, z) = (x, 0, 0) + (0, y, 0) + (0, 0, z)$$
$$= x \cdot (1, 0, 0) + y \cdot (0, 1, 0) + z \cdot (0, 0, 1)$$

Moreover, these three special vectors are “independent”

• It is a challenge to capture this intuitive notion of basis abstractly in terms of the notion of vector space

  • it can then be used for all examples of vector spaces
Independency

Definition

Vectors $v_1, \ldots, v_n$ in a vector space $V$ are called independent if for all scalars $a_1, \ldots, a_n \in \mathbb{R}$ one has:

$$a_1 \cdot v_1 + \cdots + a_n \cdot v_n = 0 \text{ in } V \text{ implies } a_1 = a_2 = \cdots = a_n = 0$$

Example

The 3 vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1) \in \mathbb{R}^3$ are independent, since if

$$a_1 \cdot (1, 0, 0) + a_2 \cdot (0, 1, 0) + a_3 \cdot (0, 0, 1) = (0, 0, 0)$$

then, using the computation from the previous slide,

$$(a_1, a_2, a_3) = (0, 0, 0), \text{ so that } a_1 = a_2 = a_3 = 0$$
Investigate (in)dependence of \[
\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}
\]

Thus we ask: are there any non-zero \(a_1, a_2, a_3 \in \mathbb{R}\) with:

\[
a_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

If there is a non-zero solution, the vectors are dependent, and if \(a_1 = a_2 = a_3 = 0\) is the only solution, they are independent.
Our question involves the systems of equations / matrix:

\[
\begin{align*}
    a_1 + 2a_2 &= 0 \\
    2a_1 - a_2 + 5a_3 &= 0 \\
    3a_1 + 4a_2 + 2a_3 &= 0
\end{align*}
\]

corresponding to

\[
\begin{pmatrix}
    1 & 2 & 0 \\
    2 & -1 & 5 \\
    3 & 4 & 2
\end{pmatrix}
\]

This one has non-zero solutions, for example:

\[
a_1 = -2, \ a_2 = 1, \ a_3 = 1
\]

(compute and check for yourself!)

Thus the original vectors are dependent. Explicitly:

\[
-2 \begin{pmatrix}
    1 \\
    2 \\
    3
\end{pmatrix} + \begin{pmatrix}
    2 \\
    -1 \\
    4
\end{pmatrix} + \begin{pmatrix}
    0 \\
    5 \\
    2
\end{pmatrix} = \begin{pmatrix}
    0 \\
    0 \\
    0
\end{pmatrix}
\]
• Same (in)dependence question for: \[
\begin{pmatrix}
1 \\
2 \\
-3
\end{pmatrix}, \begin{pmatrix}
-2 \\
1 \\
1
\end{pmatrix}, \begin{pmatrix}
1 \\
1 \\
-2
\end{pmatrix}
\]
• With corresponding matrix:
\[
\begin{pmatrix}
1 & -2 & 1 \\
2 & 1 & -1 \\
-3 & 1 & -2
\end{pmatrix}
\]
reducing to \[
\begin{pmatrix}
5 & 0 & -1 \\
0 & 5 & -3 \\
0 & 0 & -4
\end{pmatrix}
\]
• Thus the only solution is \(a_1 = a_2 = a_3 = 0\). The vectors are independent!
Non-independence / dependence

- In the plane two vectors $v, w \in \mathbb{R}^2$ are dependent if and only if:
  - they are on the same line
  - that is: $v = a \cdot w$, for some scalar $a$
- **Example:** for $v = (1, 2)$ and $w = (-2, -4)$ we have:
  - $v = -\frac{1}{2}w$, so they are on the same line
  - $a_1 \cdot v + a_2 \cdot w = 0$, e.g. for $a_1 = 2 \neq 0$ and $a_2 = 1 \neq 0$.
- In space, three vectors $u, v, w \in \mathbb{R}^3$ are dependent if they are in the same plane (or even line).
- One can prove: $v_1, \ldots, v_n \in V$ are dependent, if and only if some $v_i$ can be expressed as a linear combination of the others (the $v_j$ with $j \neq i$).
Definition

Vectors $v_1, \ldots, v_n \in V$ form a basis for a vector space $V$ if these $v_1, \ldots, v_n$
- are independent, and
- span $V$ in the sense that each $w \in V$ can be written as linear combination of these $v_1, \ldots, v_n$, namely as:

$$w = a_1 v_1 + \cdots + a_n v_n$$

for certain $a_1, \ldots, a_n \in \mathbb{R}$

- These scalars $a_i$ are uniquely determined by $w \in V$ (see below)
- A space $V$ may have several bases, but the number of elements of a basis for $V$ is always the same; it is called the dimension of $V$, usually written as $\dim(V) \in \mathbb{N}$. 
The standard basis for $\mathbb{R}^n$

For the space $\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$ there is a standard choice of base vectors:

$$(1, 0, 0 \ldots, 0), \quad (0, 1, 0, \ldots, 0), \quad \cdots \quad (0, \ldots, 0, 1)$$

We have already seen that they are independent; it is easy to see that they span $\mathbb{R}^n$.

This enables us to state precisely that $\mathbb{R}^n$ has $n$ dimensions.
An alternative basis for $\mathbb{R}^2$

- The standard basis for $\mathbb{R}^2$ is $(1, 0), (0, 1)$.
- But many other choices are possible, eg. $(1, 1), (1, -1)$
  - independence: if $a \cdot (1, 1) + b \cdot (1, -1) = (0, 0)$, then:
    \[
    \begin{cases}
    a + b = 0 \\
    a - b = 0
    \end{cases}
    \quad \text{and thus} \quad
    \begin{cases}
    a = 0 \\
    b = 0
    \end{cases}
    \]
  - spanning: each point $(x, y)$ can written in terms of $(1, 1), (1, -1)$, namely:
    \[
    (x, y) = \frac{x+y}{2} (1, 1) + \frac{x-y}{2} (1, -1)
    \]
**Theorem**

- Suppose \( V \) is a vector space, with basis \( v_1, \ldots, v_n \)
- Assume \( x \in V \) can be represented in two ways:

\[
x = a_1 v_1 + \cdots + a_n v_n \quad \text{and also} \quad x = b_1 v_1 + \cdots + b_n v_n
\]

Then: \( a_1 = b_1 \) and \( \ldots \) and \( a_n = b_n \).

**Proof:** This follows from independence of \( v_1, \ldots, v_n \) since:

\[
0 = x - x = (a_1 v_1 + \cdots + a_n v_n) - (b_1 v_1 + \cdots + b_n v_n)
= (a_1 - b_1) v_1 + \cdots + (a_n - b_n) v_n.
\]

Hence \( a_i - b_i = 0 \), by independence, and thus \( a_i = b_i \).
Functions

- A **function** $f$ is an operation that sends elements of one set $X$ to another set $Y$.
  - in that case we write $f : X \to Y$ or sometimes $X \xrightarrow{f} Y$
  - this $f$ sends $x \in X$ to $f(x) \in Y$
  - $X$ is called the **domain** and $Y$ the **codomain** of the function $f$
- Example $f(n) = \frac{1}{n+1}$ can be seen as function $\mathbb{N} \to \mathbb{Q}$, that is from the *natural* numbers $\mathbb{N}$ to the *rational* numbers $\mathbb{Q}$
- A function is sometimes also called a **map** or a **mapping**
- On each set $X$ there is the **identity** function $\text{id} : X \to X$ that does nothing: $\text{id}(x) = x$.
- Also one can compose 2 functions $X \xrightarrow{f} Y \xrightarrow{g} Z$ to a function:

$$g \circ f : X \longrightarrow Z \quad \text{given by} \quad (g \circ f)(x) = g(f(x))$$
We have seen that the two relevant operations of a vector space are addition and scalar multiplication. A linear function is required to preserve these two.

**Definition**

Let $V, W$ be two vector spaces, and $f : V \rightarrow W$ a function between them; $f$ is called linear if it preserves both:

- **addition**: for all $v, v' \in V$,
  \[
  f(v + v') = f(v) + f(v')
  \]
  in $V$ and $W$

- **scalar multiplication**: for each $v \in V$ and $a \in \mathbb{R}$,
  \[
  f(a \cdot v) = a \cdot f(v)
  \]
  in $V$ and $W$
Lemma

Each linear map $f : V \rightarrow W$ preserves:

- **zero**: $f(0) = 0$.
- **minus**: $f(-v) = -f(v)$

**Proof**: Nice illustration of axiomatic reasoning:

\[
\begin{align*}
    f(0) &= f(0) + 0 \\
    &= f(0) + (f(0) - f(0)) \\
    &= (f(0) + f(0)) - f(0) \\
    &= f(0 + 0) - f(0) \\
    &= f(0) - f(0) \\
    &= 0 \\
\end{align*}
\]

\[
\begin{align*}
    f(-v) &= f(-v) + 0 \\
    &= f(-v) + (f(v) - f(v)) \\
    &= (f(-v) + f(v)) - f(v) \\
    &= f(-v + v) - f(v) \\
    &= f(0) - f(v) \\
    &= 0 - f(v) \\
    &= -f(v) \\
\end{align*}
\]
Linear map examples I

First we consider functions $f : \mathbb{R} \rightarrow \mathbb{R}$. Most of them are not linear, like, for instance:

- $f(x) = 1 + x$, since $f(0) = 1 \neq 0$
- $f(x) = x^2$, since $f(-1) = 1 = f(1) \neq -f(1)$.

So: linear maps $\mathbb{R} \rightarrow \mathbb{R}$ can only be very simple.

**Lemma**

*Each linear map $f : \mathbb{R} \rightarrow \mathbb{R}$ is of the form $f(x) = c \cdot x$, for some $c \in \mathbb{R}$ (this constant $c$ depends on $f$)*

**Proof:** Scalar multiplication on $\mathbb{R}$ is ordinary multiplication. Hence:

$$f(x) = f(x \cdot 1) = x \cdot f(1) = f(1) \cdot x = c \cdot x, \quad \text{for } c = f(1).$$
Consider the map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by
\[
f(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3)
\]
We show in detail that this $f$ is linear, following the definition.

**Preservation of scalar multiplication** (from $\mathbb{R}^3$ to $\mathbb{R}^2$):
\[
f\left(a \cdot (x_1, x_2, x_3)\right) = f\left(a \cdot x_1, a \cdot x_2, a \cdot x_3\right)
= \left(a \cdot x_1 - a \cdot x_2, a \cdot x_2 + a \cdot x_3\right)
= \left(a \cdot (x_1 - x_2), a \cdot (x_2 + x_3)\right)
= a \cdot (x_1 - x_2, x_2 + x_3)
= a \cdot f(x_1, x_2, x_3).
\]
**Preservation of addition** of \( f \) from \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \) given by:

\[
f(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3)
\]

\[
f((x_1, x_2, x_3) + (y_1, y_2, y_3))
\]

\[
= f(x_1 + y_1, x_2 + y_2, x_3 + y_3)
\]

\[
= (x_1 + y_1 - (x_2 + y_2), (x_2 + y_2) + (x_3 + y_3))
\]

\[
= (x_1 - x_2) + (y_1 - y_2), (x_2 + x_3) + (y_2 + y_3)
\]

\[
= (x_1 - x_2, x_2 + x_3) + (y_1 - y_2, y_2 + y_3)
\]

\[
= f(x_1, x_2, x_3) + f(y_1, y_2, y_3).
\]
Consider the map \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) given by

\[
f(x, y) = (x \cos(\varphi) - y \sin(\varphi), x \sin(\varphi) + y \cos(\varphi))
\]

This map describes rotation in the plane, with angle \( \varphi \):

In the same way one can show that \( f \) is linear [Do it yourself!]
• Recall the linear map \( f(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3) \)

• **Claim**: this map is entirely determined by what it does on the base vectors \((1, 0, 0),(0, 1, 0),(0, 0, 1) \in \mathbb{R}^3\), namely:

\[
\begin{align*}
f(1, 0, 0) &= (1, 0) \\
f(0, 1, 0) &= (-1, 1) \\
f(0, 0, 1) &= (0, 1).
\end{align*}
\]

• Indeed, using linearity:

\[
\begin{align*}
f(x_1, x_2, x_3) \\
&= f\left((x_1, 0, 0) + (0, x_2, 0) + (0, 0, x_3)\right) \\
&= f\left(x_1 \cdot (1, 0, 0) + x_2 \cdot (0, 1, 0) + x_3 \cdot (0, 0, 1)\right) \\
&= f\left(x_1 \cdot (1, 0, 0)\right) + f\left(x_2 \cdot (0, 1, 0)\right) + f\left(x_3 \cdot (0, 0, 1)\right) \\
&= x_1 \cdot f(1, 0, 0) + x_2 \cdot f(0, 1, 0) + x_3 \cdot f(0, 0, 1) \\
&= x_1 \cdot (1, 0) + x_2 \cdot (-1, 1) + x_3 \cdot (0, 1) \\
&= (x_1 - x_2, x_2 + x_3)
\end{align*}
\]
Our \( f(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3) \) is thus determined by:
\[
\begin{align*}
    f(1, 0, 0) &= (1, 0) \\
    f(0, 1, 0) &= (-1, 1) \\
    f(0, 0, 1) &= (0, 1)
\end{align*}
\]

We can organise these data in a \( 2 \times 3 \) matrix:
\[
\begin{pmatrix}
    1 & -1 & 0 \\
    0 & 1 & 1
\end{pmatrix}
\]

The \( f(v_i) \), for base vector \( v_i \), appears as the \( i \)-the column.

Applying \( f \) can be done by a new kind of multiplication:
\[
\begin{pmatrix}
    1 & -1 & 0 \\
    0 & 1 & 1
\end{pmatrix} \cdot \begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix}
    1 \cdot x_1 + -1 \cdot x_2 + 0 \cdot x_3 \\
    0 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3
\end{pmatrix} = \begin{pmatrix}
    x_1 - x_2 \\
    x_2 + x_3
\end{pmatrix}
\]
The general case

The aim is to obtain a matrix for an arbitrary linear map.

- Assume a linear map \( f : V \rightarrow W \), where:
  - the vector space \( V \) has basis \( \{v_1, \ldots, v_n\} \subseteq V \);
  - \( W \) has basis \( \{w_1, \ldots, w_m\} \)
- Each \( x \in V \) can be written as \( x = a_1v_1 + \cdots + a_nv_n \). Hence:
  \[
  f(x) = f(a_1v_1 + \cdots + a_nv_n) = a_1f(v_1) + \cdots + a_nf(v_n) \quad \text{by linearity of } f
  \]
  Thus, \( f \) is determined by its values \( f(v_1), \ldots, f(v_n) \) on base vectors \( v_j \in V \).
- By writing \( f(v_j) = b_{1j}w_1 + \cdots + b_{mj}w_m \) we obtain an \( m \times n \) matrix with entries \( (b_{ij})_{i \leq m, j \leq n} \).
Towards matrix-vector multiplication

In this setting, we have:

\[ f(x) = f(a_1 v_1 + \cdots + a_n v_n) \]
\[ = a_1 f(v_1) + \cdots + a_n f(v_n) \]
\[ = a_1 (b_{11} w_1 + \cdots + b_{m1} w_m) + \cdots + a_n (b_{1n} w_1 + \cdots + b_{mn} w_m) \]
\[ = \left( a_1 b_{11} + \cdots + a_n b_{1n} \right) w_1 + \cdots + \left( a_1 b_{m1} + \cdots + a_n b_{mn} \right) w_m \]
\[ = \left( b_{11} a_1 + \cdots + b_{1n} a_n \right) w_1 + \cdots + \left( b_{m1} a_1 + \cdots + b_{mn} a_n \right) w_m \]

This motivates the definition of matrix-vector multiplication:

\[
\begin{pmatrix}
  b_{11} & \cdots & b_{1n} \\
  \vdots & \ddots & \vdots \\
  b_{m1} & \cdots & b_{mn}
\end{pmatrix}
\begin{pmatrix}
  a_1 \\
  \vdots \\
  a_n
\end{pmatrix}
= \begin{pmatrix}
  b_{11} a_1 + \cdots + b_{1n} a_n \\
  \vdots \\
  b_{m1} a_1 + \cdots + b_{mn} a_n
\end{pmatrix}
\]
Matrix-vector multiplication, concretely

- Recall $f(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3)$ with matrix:
  $$
  \begin{pmatrix}
  1 & -1 & 0 \\
  0 & 1 & 1
  \end{pmatrix}
  $$

- We can directly calculate
  $f(1, 2, -1) = (1 - 2, 2 - 1) = (-1, 1)$

- We can also get the same result by matrix-vector multiplication:
  $$
  \begin{pmatrix}
  1 & -1 & 0 \\
  0 & 1 & 1
  \end{pmatrix} \cdot \begin{pmatrix}
  1 \\
  2 \\
  -1
  \end{pmatrix} = \begin{pmatrix}
  1 \cdot 1 + -1 \cdot 2 + 0 \cdot -1 \\
  0 \cdot 1 + 1 \cdot 2 + 1 \cdot -1
  \end{pmatrix} = \begin{pmatrix}
  -1 \\
  1
  \end{pmatrix}
  $$

- This multiplication can be understood as: putting the argument values $x_1 = 1, x_2 = 2, x_3 = -1$ in variables of the underlying equations, and computing the outcome.
Another example, to learn the mechanics

\[
\begin{pmatrix}
9 & 3 & 2 & 9 & 7 \\
8 & 5 & 6 & 6 & 3 \\
4 & 5 & 8 & 9 & 3 \\
3 & 4 & 3 & 3 & 4 \\
\end{pmatrix}
\cdot
\begin{pmatrix}
9 \\
5 \\
2 \\
5 \\
7 \\
\end{pmatrix}
= 
\begin{pmatrix}
9 \cdot 9 + 3 \cdot 5 + 2 \cdot 2 + 9 \cdot 5 + 7 \cdot 7 \\
8 \cdot 9 + 5 \cdot 5 + 6 \cdot 2 + 6 \cdot 5 + 3 \cdot 7 \\
4 \cdot 9 + 5 \cdot 5 + 8 \cdot 2 + 9 \cdot 5 + 3 \cdot 7 \\
3 \cdot 9 + 4 \cdot 5 + 3 \cdot 2 + 3 \cdot 5 + 4 \cdot 7 \\
\end{pmatrix}
= 
\begin{pmatrix}
81 + 15 + 4 + 45 + 49 \\
72 + 25 + 12 + 30 + 21 \\
36 + 25 + 16 + 45 + 21 \\
27 + 20 + 6 + 15 + 28 \\
\end{pmatrix}
= 
\begin{pmatrix}
194 \\
160 \\
143 \\
96 \\
\end{pmatrix}
\]
• We have seen how a linear function can be described via a matrix
• One can also read each matrix as a linear function

Example

• Consider the matrix \[
\begin{pmatrix}
2 & 0 & -1 \\
5 & 1 & -3 \\
\end{pmatrix}
\]
• It has 3 columns/inputs and two rows/outputs. Hence it describes a function \( f : \mathbb{R}^3 \to \mathbb{R}^2 \)
• Namely: \( f(x_1, x_2, x_3) = (2x_1 - x_3, 5x_1 + x_2 - 3x_3) \).
Projections are linear maps. Consider $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

$f$ maps 3d space to the the 2d plane.

The matrix of $f$ is the following $2 \times 3$ matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$
Examples of linear maps and matrices II

We have already seen: Rotation over an angle $\varphi$ is a linear map

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \text{ given by } f(x, y) = \left( x \cos(\varphi) - y \sin(\varphi), \ x \sin(\varphi) + y \cos(\varphi) \right)$$

This rotation is described by $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = \left( x \cos(\varphi) - y \sin(\varphi), \ x \sin(\varphi) + y \cos(\varphi) \right)$$

The matrix that describes $f$ is

$$\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}.$$
Examples of linear maps and matrices III

Reflection through an axis is a linear map

- Reflection through the $y$-axis: $(x, y) \mapsto (-x, y)$ is given by

$$
\begin{pmatrix}
-1 & 0 \\
0 & 1 \\
\end{pmatrix}.
$$

- Reflection in a different straight line that goes through $(0, 0)$, for example the line $y = 2x$:
  - We first choose a different basis $E$ for $\mathbb{R}^2$, with one vector orthogonal to the axis and one on the axis.
  - We choose $E = \{(2, -1), (1, 2)\}$.
  - In terms of the basis $E$, the matrix for $f$ is just

$$
\begin{pmatrix}
-1 & 0 \\
0 & 1 \\
\end{pmatrix}.
$$

- We will learn how to transform this back to a matrix for the standard basis!
Matrix summary

- Assume bases \( \{v_1, \ldots, v_n\} \subseteq V \) and \( \{w_1, \ldots, w_m\} \subseteq W \)
- Each linear map \( f : V \to W \) corresponds to an \( m \times n \) matrix, and vice-versa.
  We often write the matrix of \( f \) as \( M_f \)
- The \( i \)-th column in this matrix \( M_f \) is given by the coefficients of \( f(v_i) \), wrt. the basis \( w_1, \ldots, w_m \) of \( W \)
- Matrix-vector multiplication corresponds to function application: \( f(v) \) is the same as \( M_f \cdot v \).
- This matrix \( M_f \) of \( f \) depends on the choice of basis: for different bases of \( V \) and \( W \) a different matrix is obtained
- (Matrix-vector multiplication forms itself a linear function)
The identity matrix

Consider the following $n \times n$ identity matrix with diagonal of 1’s:

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- To which function does $I_n$ correspond? The identity function $\mathbb{R}^n \rightarrow \mathbb{R}^n$.
- To which system of equations does $I_n$ correspond?

$$\begin{cases} x_1 = 0 \\ \vdots \\ x_n = 0 \end{cases}$$
Matrices as vectors I

- Write $\text{Mat}_{m,n} = \{ M \mid M \text{ is an } m \times n \text{ matrix} \}$
- Thus each $M \in \text{Mat}_{m,n}$ can be written as $M = (a_{ij})$, for $1 \leq i \leq m$ and $1 \leq j \leq n$
- We can add two such matrices $M, N \in \text{Mat}_{m,n}$, giving $M + N \in \text{Mat}_{m,n}$.
  - the matrices are added entry-wise, that is:
  - if $M = (a_{ij}), N = (b_{ij}), M + N = (c_{ij})$, then $c_{ij} = a_{ij} + b_{ij}$
- Similarly, matrices can be multiplied by a scalar $s \in \mathbb{R}$
  - $s \cdot M \in \text{Mat}_{m,n}$ has entries $s \cdot a_{ij}$
- Finally, there is a zero matrix $0_{m,n} \in \text{Mat}_{m,n}$, with only zeros as entries

$\text{Mat}_{m,n}$ is a vector space
Matrices as vectors II: example

- **Addition:**

\[
\begin{pmatrix}
2 & 0 & 1 \\
-1 & -3 & 5 \\
\end{pmatrix} +
\begin{pmatrix}
1 & 1 & 2 \\
2 & -2 & 5 \\
\end{pmatrix} =
\begin{pmatrix}
3 & 1 & 3 \\
1 & -5 & 10 \\
\end{pmatrix}
\]

- **Scalar multiplication:**

\[
5 \cdot \begin{pmatrix}
2 & 0 & 1 \\
-1 & -3 & 5 \\
\end{pmatrix} =
\begin{pmatrix}
10 & 0 & 5 \\
-5 & -15 & 25 \\
\end{pmatrix}
\]
Matrices as vectors III: transpose

- For a matrix $M \in \text{Mat}_{m,n}$ write $M^T \in \text{Mat}_{n,m}$ for the transpose of $M$.
- It is obtained by mirroring:
  - if $M = (a_{ij})$ then $M^T$ has entries $a_{ji}$
  - For example

$$\begin{pmatrix} 2 & 0 & 1 \\ -1 & -3 & 5 \end{pmatrix}^T = \begin{pmatrix} 2 & -1 \\ 0 & -3 \\ 1 & 5 \end{pmatrix}$$

**Theorem**

Transposition is a linear map $(-)^T : \text{Mat}_{m,n} \to \text{Mat}_{n,m}$. That is:

- $(M + N)^T = M^T + N^T$
- $(a \cdot M)^T = a \cdot M^T$