Exercises on Lecture 10
Higher order logic and the Calculus of Constructions

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NB All exercises can also be made with Coq. Please look at the web site for the `.v` file. You can create a new file under your proofweb login and work with it.

1. Definition in CC (for $t, q : A$):

$$t =_A q := \Pi P{:}A{\to}{*}.(Pt \to Pq)$$

   - (basic) Prove that this equality is reflexive and transitive by giving terms of the types $\Pi x{:}A.x{=}_A x$ and of $\Pi x, y, z{:}A.x{=}_A y \to y{=}_A z \to x{=}_A z$
     ANSWER:
     $\mathsf{refl} := \lambda x{:}A.\lambda P{:}A{\to}*.\lambda h{:}P\,x.h$
     $\quad : \Pi x{:}A.x{=}_A x$
     $\lambda x, y, z{:}A.\lambda f{:}x{=}_A y.\lambda g{:}y{=}_A z.\lambda P{:}A{\to}*.\lambda h{:}P\,x.g\,P(f\,P\,h)$
     $\quad : \Pi x, y, z{:}A.x{=}_A y \to y{=}_A z \to x{=}_A z$
   - (advanced) Prove that this equality is symmetric by giving a term of the type $\Pi x, y{:}A.x{=}_A y \to y{=}_A x$.
     ANSWER: (Here are three answers. Check for yourself that each of these terms is well-typed.)
     $\lambda x, y{:}A.\lambda f{:}x{=}_A y.f\,(\lambda z{:}A.z{=}_A x)(\mathsf{refl}\,x)$
     $\quad : \Pi x, y, z{:}A.x{=}_A y \to y{=}_A x$
     $\lambda x, y{:}A.\lambda f{:}x{=}_A y.\lambda P{:}A{\to}*.f\,(\lambda z{:}A.P\,z{\to}P\,x)(\lambda q{:}P\,x.q)$
     $\quad : \Pi x, y, z{:}A.x{=}_A y \to y{=}_A x$
     $\lambda x, y{:}A.\lambda f{:}x{=}_A y.\lambda P{:}A{\to}*\,\lambda h{:}P\,y.f\,(\lambda z{:}A.P\,z{\to}P\,x)(\lambda q{:}P\,x.q)h$
     $\quad : \Pi x, y, z{:}A.x{=}_A y \to y{=}_A x$

2. The transitive closure of a binary relation R on A has been defined as follows.

$$\begin{array}{lcl}\mathsf{trclos}\,R & := & \lambda x, y{:}A.\\ & & (\forall Q{:}A{\to}A{\to}*.(\mathsf{trans}(Q){\to}(R \subseteq Q){\to}(Q\,x\,y))).\end{array}$$

- (basic) Prove that the transitive closure of R contains R.
  ANSWER: See the Coq file `Exercise10_inCoq.v`

```
fun (R : A -> A -> Prop) (x y : A) (H : R x y) (Q : A -> A -> Prop)
  (_ : trans Q) (Hsub : subset R Q) => Hsub x y H
     : forall R : A -> A -> Prop, subset R (transclos R)
```

- (medium) Prove that the transitive closure is transitive.

3. The existential quantifier has been defined by

$$\exists x{:}\sigma.\phi := \forall \alpha{:}\ast.(\forall x{:}\sigma.\phi \to \alpha) \to \alpha$$

- (medium) Given $t : \sigma$ and $q : P\,t$, give a term M such that $M : \exists x{:}\sigma.P\,x$
- (medium) Given $q : \exists x{:}\sigma.P\,x$ and $h : \forall y{:}\sigma.P\,y \to C$ with $y \notin \mathrm{FV}(C)$, give a term N of type C.