

# Exercises on the Lecture

## Higher order logic and the Calculus of Constructions

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NB All exercises can also be made with Coq. Please look at the web site for the .v file. You can create a new file under your proofweb login and work with it.

1. Definition in CC (for  $t, q : A$ ):

$$t =_A q := \Pi P : A \rightarrow *. (Pt \rightarrow Pq)$$

- (basic) Prove that this equality is reflexive and transitive by giving terms of the types  $\Pi x : A. x =_A x$  and of  $\Pi x, y, z : A. x =_A y \rightarrow y =_A z \rightarrow x =_A z$

ANSWER:

$$\lambda x : A. \lambda P : A \rightarrow *. \lambda h : P x. h : \Pi x : A. x =_A x$$

$$\lambda x, y, z : A. \lambda f : x =_A y. \lambda g : y =_A z. \lambda P : A \rightarrow *. \lambda h : P x. g P (f P h) : \Pi x, y, z : A. x =_A y \rightarrow y =_A z \rightarrow x =_A z$$

- (advanced) Prove that this equality is symmetric by giving a term of the type  $\Pi x, y : A. x =_A y \rightarrow y =_A x$ .

2. The transitive closure of a binary relation  $R$  on  $A$  has been defined as follows.

$$\text{trclos } R := \lambda x, y : A.$$

$$(\forall Q : A \rightarrow A \rightarrow *. (\text{trans}(Q) \rightarrow (R \subseteq Q) \rightarrow (Q x y))).$$

- (basic) Prove that the transitive closure of  $R$  contains  $R$ .
- (medium) Prove that the transitive closure is transitive.

3. The existential quantifier has been defined by

$$\exists x : \sigma. \phi := \forall \alpha : *. (\forall x : \sigma. \phi \rightarrow \alpha) \rightarrow \alpha$$

- (medium) Given  $t : \sigma$  and  $q : Pt$ , give a term  $M$  such that  $M : \exists x : \sigma. Px$
- (medium) Given  $q : \exists x : \sigma. Px$  and  $h : \forall y : \sigma. Py \rightarrow C$  with  $y \notin \text{FV}(C)$ , give a term  $N$  of type  $C$ .