Proving with Computer Assistance, 2IM15

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Exercises on Lecture: Simple Type theory and *Formulas-as-Types* for propositional logic (Some Answers)

See the course notes – notably *Introduction to Type Theory* by Herman Geuvers – and the slides on the homepage.

1. Verify in detail (by giving a derivation in $\lambda\!\!\to$) that

$$\lambda x^{\alpha\to\beta}.\lambda y^{\beta\to\gamma}.\lambda z^{\alpha}.y(xz) : (\alpha\to\beta)\to(\beta\to\gamma)\to\alpha\to\gamma$$

2. (a) Verify in detail (by giving a derivation in $\lambda\!\!\to$) that

$$\lambda x^{\beta\to\alpha}.\lambda y^{(\beta\to\alpha)\to\alpha}.y(\lambda z^{\beta}.x\, z) : (\beta\to\alpha)\to((\beta\to\alpha)\to\alpha)\to\alpha$$

   (b) "Dress up" the λ-term $\lambda x.\lambda y.y(\lambda z.x\, z)$ with type information in such a way that it is of type $(\beta\to\gamma)\to((\beta\to\gamma)\to\alpha)\to\alpha$

   (c) Give a "simpler" term of type $(\beta\to\gamma)\to((\beta\to\gamma)\to\alpha)\to\alpha$.

3. Give the natural deduction (either in Fitch style or in tree form) that corresponds to

$$\lambda x{:}\gamma\to\varepsilon.\lambda y{:}(\gamma\to\varepsilon)\to\varepsilon.y(\lambda z{:}\gamma.y\, x) : (\gamma\to\varepsilon)\to((\gamma\to\varepsilon)\to\varepsilon)\to\varepsilon$$

4. Give another term of the same type

$$(\gamma\to\varepsilon)\to((\gamma\to\varepsilon)\to\varepsilon)\to\varepsilon$$

   and the natural deduction (either in Fitch style or in tree form) that it corresponds to.

5. In all of the following cases: give a typing derivation.

   (a) Find a term of type $(\delta\to\delta\to\alpha)\to(\alpha\to\beta\to\gamma)\to(\delta\to\beta)\to\delta\to\gamma$
   ANSWER: Finding a term is best done by finding a derivation of (a term of) this type as a formula. Call $\sigma := (\delta\to\delta\to\alpha)\to(\alpha\to\beta\to\gamma)\to(\delta\to\beta)\to\delta\to\gamma$

1	$x : \delta{\to}\delta{\to}\alpha$	
2	$y : \alpha{\to}\beta{\to}\gamma$	
3	$z : \delta{\to}\beta$	
4	$v : \delta$	
5	$x\,v : \delta{\to}\alpha$	app, 1, 4
6	$x\,v\,v : \alpha$	app, 5, 4
7	$y(x\,v\,v) : \beta{\to}\gamma$	app, 2, 6
8	$z\,v) : \beta$	app, 3, 4
9	$y(x\,v\,v)(z\,v) : \gamma$	app, 2, 6
10	$\lambda v : \delta.y(x\,v\,v)(z\,v) : \delta{\to}\gamma$	λ-rule, 4, 9
11	$\lambda z : \delta{\to}\beta.\lambda v : \delta.y(x\,v\,v)(z\,v) : (\delta{\to}\beta){\to}\delta{\to}\gamma$	λ-rule, 3, 10
12	$\lambda y : \alpha{\to}\beta{\to}\gamma.\lambda z : \delta{\to}\beta.\lambda v : \delta.y(x\,v\,v)(z\,v) : (\alpha{\to}\beta{\to}\gamma){\to}(\delta{\to}\beta){\to}\delta{\to}\gamma$	λ-rule, 2, 11
13	$\lambda x : \delta{\to}\delta{\to}\alpha.\lambda y : \alpha{\to}\beta{\to}\gamma.\lambda z : \delta{\to}\beta.\lambda v : \delta.y(x\,v\,v)(z\,v) : \sigma$	λ-rule, 1, 12

This term is created by filling in the ? in the following "template"

1	$x : \delta{\to}\delta{\to}\alpha$	
2	$y : \alpha{\to}\beta{\to}\gamma$	
3	$z : \delta{\to}\beta$	
4	$v : \delta$	
5	$\ldots$	
6	$\ldots$	
7	$\ldots$	
8	$\ldots$	
9	$? : \gamma$	
10	$\lambda v : \delta.? : \delta{\to}\gamma$	λ-rule, 4, .
11	$\lambda z : \delta{\to}\beta.\lambda v : \delta.? : (\delta{\to}\beta){\to}\delta{\to}\gamma$	λ-rule, 3, .
12	$\lambda y : \alpha{\to}\beta{\to}\gamma.\lambda z : \delta{\to}\beta.\lambda v : \delta.? : (\alpha{\to}\beta{\to}\gamma){\to}(\delta{\to}\beta){\to}\delta{\to}\gamma$	λ-rule, 2, .
13	$\lambda x : \delta{\to}\delta{\to}\alpha.\lambda y : \alpha{\to}\beta{\to}\gamma.\lambda z : \delta{\to}\beta.\lambda v : \delta.? : \sigma$	λ-rule, 1, .

The $? : \gamma$ should clearly be of the form $y?_1?_2$ with $?_1 : \alpha$ and $?_2 : \beta$ ... and so forth. So one basically works "inside out" constructing the term. (This is basically "goal directed theorem proving".)

(b) Find two terms of type $(\delta{\to}\delta{\to}\alpha){\to}(\gamma{\to}\alpha){\to}(\alpha{\to}\beta){\to}\delta{\to}\gamma{\to}\beta$
ANSWER: (Construct the derivations yourself.)
$\lambda x : \delta{\to}\delta{\to}\alpha.\lambda y : \gamma{\to}\alpha.\lambda z : \alpha{\to}\beta.\lambda v : \delta.\lambda w : \gamma.z\,(x\,v\,v)$
$\lambda x : \delta{\to}\delta{\to}\alpha.\lambda y : \gamma{\to}\alpha.\lambda z : \alpha{\to}\beta.\lambda v : \delta.\lambda w : \gamma.z\,(y\,w)$

(c) Find a term of type $((\alpha\to\beta)\to\alpha)\to(\alpha\to\alpha\to\beta)\to\alpha$
ANSWER: (Construct the derivation yourself.)
$\lambda f : (\alpha\to\beta)\to\alpha.\lambda g : \alpha\to\alpha\to\beta.f(\lambda x{:}\alpha.g\,x\,x)$

(d) Find a term of type $((\alpha\to\beta)\to\alpha)\to(\alpha\to\alpha\to\beta)\to\beta$
ANSWER: We use the previous exercise! (Construct the derivation yourself.)
$\lambda f : (\alpha\to\beta)\to\alpha.\lambda g : \alpha\to\alpha\to\beta.g(f(\lambda x{:}\alpha.g\,x\,x))(f(\lambda x{:}\alpha.g\,x\,x))$.