Proving with Computer Assistance, 2IMF15

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Exercises on Lecture: Simple Type theory and *Formulas-as-Types* for propositional logic

See the course notes – notably *Introduction to Type Theory* by Herman Geuvers – and the slides on the homepage.

1. Verify in detail (by giving a derivation in $\lambda{\to}$) that

$$\lambda x^{\alpha\to\beta}.\lambda y^{\beta\to\gamma}.\lambda z^{\alpha}.y(xz) : (\alpha{\to}\beta){\to}(\beta{\to}\gamma){\to}\alpha{\to}\gamma$$

2. (a) Verify in detail (by giving a derivation in $\lambda{\to}$) that

$$\lambda x^{\beta\to\alpha}.\lambda y^{(\beta\to\alpha)\to\alpha}.y(\lambda z^{\beta}.x\,z) : (\beta{\to}\alpha){\to}((\beta{\to}\alpha){\to}\alpha){\to}\alpha$$

   (b) "Dress up" the λ-term $\lambda x.\lambda y.y(\lambda z.x\,z)$ with type information in such a way that it is of type $(\beta{\to}\gamma){\to}((\beta{\to}\gamma){\to}\alpha){\to}\alpha$

   (c) Give a "simpler" term of type $(\beta{\to}\gamma){\to}((\beta{\to}\gamma){\to}\alpha){\to}\alpha$.

3. (a) Give the natural deduction (either in Fitch style or in tree form) that corresponds to

$$\lambda x{:}\gamma{\to}\varepsilon.\lambda y{:}(\gamma{\to}\varepsilon){\to}\varepsilon.y(\lambda z{:}\gamma.y\,x) : (\gamma{\to}\varepsilon){\to}((\gamma{\to}\varepsilon){\to}\varepsilon){\to}\varepsilon$$

   (b) Give another term of the same type

$$(\gamma{\to}\varepsilon){\to}((\gamma{\to}\varepsilon){\to}\varepsilon){\to}\varepsilon$$

   and the natural deduction (either in Fitch style or in tree form) that it corresponds to.

4. In all of the following cases: give a typing derivation.

   (a) Find a term of type $(\delta{\to}\delta{\to}\alpha){\to}(\alpha{\to}\beta{\to}\gamma){\to}(\delta{\to}\beta){\to}\delta{\to}\gamma$

   (b) Find two terms of type $(\delta{\to}\delta{\to}\alpha){\to}(\gamma{\to}\alpha){\to}(\alpha{\to}\beta){\to}\delta{\to}\gamma{\to}\beta$

   (c) Find a term of type $((\alpha{\to}\beta){\to}\alpha){\to}(\alpha{\to}\alpha{\to}\beta){\to}\alpha$

   (d) Find a term of type $((\alpha{\to}\beta){\to}\alpha){\to}(\alpha{\to}\alpha{\to}\beta){\to}\beta$ (Hint: use the previous exercise.)

5. Consider the following term "with holes" N, where $A = \alpha{\to}\alpha$ and $\mathbf{I}_1$ and $\mathbf{I}_2$ and $\mathbf{I}_3$ and $\mathbf{I}_4$ are copies of the well-known λ-term $\mathbf{I}$ $(:= \lambda x.x)$.

$$N \quad := \quad \lambda y:?.\,(\lambda x{:}A{\to}A.\,\mathbf{I}_1\,(x\,\mathbf{I}_4\,(\mathbf{I}_3\,y)))\,\mathbf{I}_2$$

   Fill in the type for ? in N, give the types for $\mathbf{I}_1$ and $\mathbf{I}_2$ and $\mathbf{I}_3$ and $\mathbf{I}_4$ and give the type of N itself in simple type theory $(\lambda{\to})$ à la Church. (Note that A abbreviates $\alpha{\to}\alpha$.)