Proving with Computer Assistance, 2IMF15

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Exercises on Lecture: Simple Type theory and *Formulas-as-Types* for propositional logic

See the course notes – notably *Introduction to Type Theory* by Herman Geuvers – and the slides on the homepage.

1. Verify in detail (by giving a derivation in $\lambda\!\to$) that

$$\lambda x^{\alpha\to\beta}.\lambda y^{\beta\to\gamma}.\lambda z^{\alpha}.y(xz) : (\alpha\to\beta)\to(\beta\to\gamma)\to\alpha\to\gamma$$

2. (a) Verify in detail (by giving a derivation in $\lambda\!\to$) that

$$\lambda x^{\beta\to\alpha}.\lambda y^{(\beta\to\alpha)\to\alpha}.y(\lambda z^{\beta}.x\,z) : (\beta\to\alpha)\to((\beta\to\alpha)\to\alpha)\to\alpha$$

   (b) "Dress up" the λ-term $\lambda x.\lambda y.y(\lambda z.x\,z)$ with type information in such a way that it is of type $(\beta\to\gamma)\to((\beta\to\gamma)\to\alpha)\to\alpha$
   Answer: ....
   Here is the term without typing derivation.
   $\lambda x{:}\beta\to\gamma.\lambda y{:}(\beta\to\gamma)\to\alpha.y(\lambda z{:}\beta.x\,z)$
   End Answer ....

   (c) Give a "simpler" term of type $(\beta\to\gamma)\to((\beta\to\gamma)\to\alpha)\to\alpha$.
   Answer: ....
   Here is the term without typing derivation.
   $\lambda x{:}\beta\to\gamma.\lambda y{:}(\beta\to\gamma)\to\alpha.y\,x$
   End Answer ....

3. Give the natural deduction (either in Fitch style or in tree form) that corresponds to

$$\lambda x{:}\gamma\to\varepsilon.\lambda y{:}(\gamma\to\varepsilon)\to\varepsilon.y(\lambda z{:}\gamma.y\,x) : (\gamma\to\varepsilon)\to((\gamma\to\varepsilon)\to\varepsilon)\to\varepsilon$$

4. Give another term of the same type

$$(\gamma\to\varepsilon)\to((\gamma\to\varepsilon)\to\varepsilon)\to\varepsilon$$

   and the natural deduction (either in Fitch style or in tree form) that it corresponds to.

5. In all of the following cases: give a typing derivation.

   (a) Find a term of type $(\delta\to\delta\to\alpha)\to(\alpha\to\beta\to\gamma)\to(\delta\to\beta)\to\delta\to\gamma$
   Answer: ....

Finding a term is best done by finding a derivation of (a term of) this type as a formula. Call $\sigma := (\delta\to\delta\to\alpha)\to(\alpha\to\beta\to\gamma)\to(\delta\to\beta)\to\delta\to\gamma$

1	$x : \delta\to\delta\to\alpha$	
2	$y : \alpha\to\beta\to\gamma$	
3	$z : \delta\to\beta$	
4	$v : \delta$	
5	$x\,v : \delta\to\alpha$	app, 1, 4
6	$x\,v\,v : \alpha$	app, 5, 4
7	$y(x\,v\,v) : \beta\to\gamma$	app, 2, 6
8	$z\,v) : \beta$	app, 3, 4
9	$y(x\,v\,v)(z\,v) : \gamma$	app, 2, 6
10	$\lambda v : \delta.y(x\,v\,v)(z\,v) : \delta\to\gamma$	λ-rule, 4, 9
11	$\lambda z : \delta\to\beta.\lambda v : \delta.y(x\,v\,v)(z\,v) : (\delta\to\beta)\to\delta\to\gamma$	λ-rule, 3, 10
12	$\lambda y : \alpha\to\beta\to\gamma.\lambda z : \delta\to\beta.\lambda v : \delta.y(x\,v\,v)(z\,v) : (\alpha\to\beta\to\gamma)\to(\delta\to\beta)\to\delta\to\gamma$	λ-rule, 2, 11
13	$\lambda x : \delta\to\delta\to\alpha.\lambda y : \alpha\to\beta\to\gamma.\lambda z : \delta\to\beta.\lambda v : \delta.y(x\,v\,v)(z\,v) : \sigma$	λ-rule, 1, 12

This term is created by filling in the ? in the following "template"

1	$x : \delta\to\delta\to\alpha$	
2	$y : \alpha\to\beta\to\gamma$	
3	$z : \delta\to\beta$	
4	$v : \delta$	
5	...	
6	...	
7	...	
8	...	
9	$? : \gamma$	
10	$\lambda v : \delta.? : \delta\to\gamma$	λ-rule, 4, .
11	$\lambda z : \delta\to\beta.\lambda v : \delta.? : (\delta\to\beta)\to\delta\to\gamma$	λ-rule, 3, .
12	$\lambda y : \alpha\to\beta\to\gamma.\lambda z : \delta\to\beta.\lambda v : \delta.? : (\alpha\to\beta\to\gamma)\to(\delta\to\beta)\to\delta\to\gamma$	λ-rule, 2, .
13	$\lambda x : \delta\to\delta\to\alpha.\lambda y : \alpha\to\beta\to\gamma.\lambda z : \delta\to\beta.\lambda v : \delta.? : \sigma$	λ-rule, 1, .

The $? : \gamma$ should clearly be of the form $y?_1?_2$ with $?_1 : \alpha$ and $?_2 : \beta$ ... and so forth. So one basically works "inside out" constructing the term. (This is basically "goal directed theorem proving".)

End Answer ....

(b) Find two terms of type $(\delta\to\delta\to\alpha)\to(\gamma\to\alpha)\to(\alpha\to\beta)\to\delta\to\gamma\to\beta$

Answer: ....

Here are the terms, construct the derivations yourself.

$\lambda x : \delta\to\delta\to\alpha.\lambda y : \gamma\to\alpha.\lambda z : \alpha\to\beta.\lambda v : \delta.\lambda w : \gamma.z\,(x\,v\,v)$

$\lambda x : \delta\to\delta\to\alpha.\lambda y : \gamma\to\alpha.\lambda z : \alpha\to\beta.\lambda v : \delta.\lambda w : \gamma.z\,(y\,w)$

End Answer ....

(c) Find a term of type $((\alpha\to\beta)\to\alpha)\to(\alpha\to\alpha\to\beta)\to\alpha$

Answer: ....

Here is the term, construct the derivation yourself. $\lambda f : (\alpha\to\beta)\to\alpha.\lambda g : \alpha\to\alpha\to\beta.f(\lambda x{:}\alpha.g\,x\,x)$

End Answer ....

(d) Find a term of type $((\alpha\to\beta)\to\alpha)\to(\alpha\to\alpha\to\beta)\to\beta$ (Hint: use the previous exercise.)

Answer: ....

Here is the term, construct the derivation yourself.

$\lambda f : (\alpha\to\beta)\to\alpha.\lambda g : \alpha\to\alpha\to\beta.g(f(\lambda x{:}\alpha.g\,x\,x))(f(\lambda x{:}\alpha.g\,x\,x))$.

End Answer ....