Proving with Computer Assistance, 2IMF15

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Exercises on Lecture: Simple Type theory and *Formulas-as-Types* for propositional logic

See the course notes – notably *Introduction to Type Theory* by Herman Geuvers – and the slides on the homepage.

1. Verify in detail (by giving a derivation in $\lambda\!\!\rightarrow$) that

$$\lambda x^{\alpha\to\beta}.\lambda y^{\beta\to\gamma}.\lambda z^{\alpha}.y(xz) : (\alpha\to\beta)\to(\beta\to\gamma)\to\alpha\to\gamma$$

2. (a) Verify in detail (by giving a derivation in $\lambda\!\!\rightarrow$) that

$$\lambda x^{\beta\to\alpha}.\lambda y^{(\beta\to\alpha)\to\alpha}.y(\lambda z^{\beta}.x\,z) : (\beta\to\alpha)\to((\beta\to\alpha)\to\alpha)\to\alpha$$

   (b) "Dress up" the λ-term $\lambda x.\lambda y.y(\lambda z.x\,z)$ with type information in such a way that it is of type $(\beta\to\gamma)\to((\beta\to\gamma)\to\alpha)\to\alpha$
   Answer: .....
   Here is the term without typing derivation.
   $\lambda x{:}\beta\to\gamma.\lambda y{:}(\beta\to\gamma)\to\alpha.y(\lambda z{:}\beta.x\,z)$
   End Answer .....

   (c) Give a "simpler" term of type $(\beta\to\gamma)\to((\beta\to\gamma)\to\alpha)\to\alpha$.
   Answer: .....
   Here is the term without typing derivation.
   $\lambda x{:}\beta\to\gamma.\lambda y{:}(\beta\to\gamma)\to\alpha.y\,x$
   End Answer .....

3. (a) Give the natural deduction (either in Fitch style or in tree form) that corresponds to

$$\lambda x{:}\gamma\to\varepsilon.\lambda y{:}(\gamma\to\varepsilon)\to\varepsilon.y(\lambda z{:}\gamma.y\,x) : (\gamma\to\varepsilon)\to((\gamma\to\varepsilon)\to\varepsilon)\to\varepsilon$$

   (b) Give another term of the same type

$$(\gamma\to\varepsilon)\to((\gamma\to\varepsilon)\to\varepsilon)\to\varepsilon$$

   and the natural deduction (either in Fitch style or in tree form) that it corresponds to.

4. In all of the following cases: give a typing derivation.

   (a) Find a term of type $(\delta\to\delta\to\alpha)\to(\alpha\to\beta\to\gamma)\to(\delta\to\beta)\to\delta\to\gamma$
   Answer: .....

Finding a term is best done by finding a derivation of (a term of) this type as a formula. Call $\sigma := (\delta\to\delta\to\alpha)\to(\alpha\to\beta\to\gamma)\to(\delta\to\beta)\to\delta\to\gamma$

1	$x : \delta\to\delta\to\alpha$	
2	$\quad y : \alpha\to\beta\to\gamma$	
3	$\quad\quad z : \delta\to\beta$	
4	$\quad\quad\quad v : \delta$	
5	$\quad\quad\quad x\,v : \delta\to\alpha$	app, 1, 4
6	$\quad\quad\quad x\,v\,v : \alpha$	app, 5, 4
7	$\quad\quad\quad y(x\,v\,v) : \beta\to\gamma$	app, 2, 6
8	$\quad\quad\quad z\,v) : \beta$	app, 3, 4
9	$\quad\quad\quad y(x\,v\,v)(z\,v) : \gamma$	app, 2, 6
10	$\quad\quad\lambda v : \delta.y(x\,v\,v)(z\,v) : \delta\to\gamma$	λ-rule, 4, 9
11	$\quad\lambda z : \delta\to\beta.\lambda v : \delta.y(x\,v\,v)(z\,v) : (\delta\to\beta)\to\delta\to\gamma$	λ-rule, 3, 10
12	$\lambda y : \alpha\to\beta\to\gamma.\lambda z : \delta\to\beta.\lambda v : \delta.y(x\,v\,v)(z\,v) : (\alpha\to\beta\to\gamma)\to(\delta\to\beta)\to\delta\to\gamma$	λ-rule, 2, 11
13	$\lambda x : \delta\to\delta\to\alpha.\lambda y : \alpha\to\beta\to\gamma.\lambda z : \delta\to\beta.\lambda v : \delta.y(x\,v\,v)(z\,v) : \sigma$	λ-rule, 1, 12

This term is created by filling in the ? in the following "template"

1	$x : \delta\to\delta\to\alpha$	
2	$\quad y : \alpha\to\beta\to\gamma$	
3	$\quad\quad z : \delta\to\beta$	
4	$\quad\quad\quad v : \delta$	
5	$\quad\quad\quad \ldots$	
6	$\quad\quad\quad \ldots$	
7	$\quad\quad\quad \ldots$	
8	$\quad\quad\quad \ldots$	
9	$\quad\quad\quad ? : \gamma$	
10	$\quad\quad\lambda v : \delta.? : \delta\to\gamma$	λ-rule, 4, .
11	$\quad\lambda z : \delta\to\beta.\lambda v : \delta.? : (\delta\to\beta)\to\delta\to\gamma$	λ-rule, 3, .
12	$\lambda y : \alpha\to\beta\to\gamma.\lambda z : \delta\to\beta.\lambda v : \delta.? : (\alpha\to\beta\to\gamma)\to(\delta\to\beta)\to\delta\to\gamma$	λ-rule, 2, .
13	$\lambda x : \delta\to\delta\to\alpha.\lambda y : \alpha\to\beta\to\gamma.\lambda z : \delta\to\beta.\lambda v : \delta.? : \sigma$	λ-rule, 1, .

The $? : \gamma$ should clearly be of the form $y?_1?_2$ with $?_1 : \alpha$ and $?_2 : \beta$ ... and so forth. So one basically works "inside out" constructing the term. (This is basically "goal directed theorem proving".)
End Answer .....

(b) Find two terms of type $(\delta\to\delta\to\alpha)\to(\gamma\to\alpha)\to(\alpha\to\beta)\to\delta\to\gamma\to\beta$
Answer: .....
Here are the terms, construct the derivations yourself.
$\lambda x:\delta\to\delta\to\alpha.\lambda y:\gamma\to\alpha.\lambda z:\alpha\to\beta.\lambda v:\delta.\lambda w:\gamma.z\,(x\,v\,v)$
$\lambda x:\delta\to\delta\to\alpha.\lambda y:\gamma\to\alpha.\lambda z:\alpha\to\beta.\lambda v:\delta.\lambda w:\gamma.z\,(y\,w)$
End Answer .....

(c) Find a term of type $((\alpha\to\beta)\to\alpha)\to(\alpha\to\alpha\to\beta)\to\alpha$
Answer: .....
Here is the term, construct the derivation yourself. $\lambda f:(\alpha\to\beta)\to\alpha.\lambda g:\alpha\to\alpha\to\beta.f(\lambda x{:}\alpha.g\,x\,x)$
End Answer .....

(d) Find a term of type $((\alpha\to\beta)\to\alpha)\to(\alpha\to\alpha\to\beta)\to\beta$ (Hint: use the previous exercise.)
Answer: .....
Here is the term, construct the derivation yourself.
$\lambda f:(\alpha\to\beta)\to\alpha.\lambda g:\alpha\to\alpha\to\beta.g(f(\lambda x{:}\alpha.g\,x\,x))(f(\lambda x{:}\alpha.g\,x\,x))$.
End Answer .....

5. Consider the following term "with holes" N, where $A=\alpha\to\alpha$ and $\mathbf{I}_1$ and $\mathbf{I}_2$ and $\mathbf{I}_3$ and $\mathbf{I}_4$ are copies of the well-known λ-term $\mathbf{I}$ ($:=\lambda x.x$).

$$N \quad := \quad \lambda y:?.\,(\lambda x{:}A\to A.\,\mathbf{I}_1\,(x\,\mathbf{I}_4\,(\mathbf{I}_3\,y)))\,\mathbf{I}_2$$

Fill in the type for ? in N, give the types for $\mathbf{I}_1$ and $\mathbf{I}_2$ and $\mathbf{I}_3$ and $\mathbf{I}_4$ and give the type of N itself in simple type theory ($\lambda\to$) à la Church. (Note that A abbreviates $\alpha\to\alpha$.)
Answer: .....
$\mathbf{I}_2:A\to A$ and $\mathbf{I}_4:A$ (because $\mathbf{I}_4$ is the argument of x), so $\mathbf{I}_3\,y:\alpha$ (because it is the argument of $x\,\mathbf{I}_4:A$) and hence $\mathbf{I}_3:A$ and $y:\alpha$, so $?=\alpha$. We have $x\,\mathbf{I}_4\,(\mathbf{I}_3\,y):\alpha$, so $\mathbf{I}_1:A$ and $\mathbf{I}_1\,(x\,\mathbf{I}_4\,(\mathbf{I}_3\,y)):\alpha$. The type of N is $\alpha\to\alpha$, so A.
End Answer .....