Proving with Computer Assistance, 2IF65

Herman Geuvers, TUE

Exercises on Lecture 3: Answers

On simple type theory à la Church

1. Find a term of type $(\delta{\to}\delta{\to}\alpha){\to}(\alpha{\to}\beta{\to}\gamma){\to}(\delta{\to}\beta){\to}\delta{\to}\gamma$

   ANSWER: Finding a term is best done by finding a derivation of (a term of) this type as a formula. Call $\sigma := (\delta{\to}\delta{\to}\alpha){\to}(\alpha{\to}\beta{\to}\gamma){\to}(\delta{\to}\beta){\to}\delta{\to}\gamma$

1	$x : \delta{\to}\delta{\to}\alpha$	
2	$y : \alpha{\to}\beta{\to}\gamma$	
3	$z : \delta{\to}\beta$	
4	$v : \delta$	
5	$x\,v : \delta{\to}\alpha$	app, 1, 4
6	$x\,v\,v : \alpha$	app, 5, 4
7	$y(x\,v\,v) : \beta{\to}\gamma$	app, 2, 6
8	$z\,v) : \beta$	app, 3, 4
9	$y(x\,v\,v)(z\,v) : \gamma$	app, 2, 6
10	$\lambda v : \delta.y(x\,v\,v)(z\,v) : \delta{\to}\gamma$	λ-rule, 4, 9
11	$\lambda z : \delta{\to}\beta.\lambda v : \delta.y(x\,v\,v)(z\,v) : (\delta{\to}\beta){\to}\delta{\to}\gamma$	λ-rule, 3, 10
12	$\lambda y : \alpha{\to}\beta{\to}\gamma.\lambda z : \delta{\to}\beta.\lambda v : \delta.y(x\,v\,v)(z\,v) : (\alpha{\to}\beta{\to}\gamma){\to}(\delta{\to}\beta){\to}\delta{\to}\gamma$	λ-rule, 2, 11
13	$\lambda x : \delta{\to}\delta{\to}\alpha.\lambda y : \alpha{\to}\beta{\to}\gamma.\lambda z : \delta{\to}\beta.\lambda v : \delta.y(x\,v\,v)(z\,v) : \sigma$	λ-rule, 1, 12

   This term is created by filling in the ? in the following "template"

1	$x : \delta{\to}\delta{\to}\alpha$	
2	$y : \alpha{\to}\beta{\to}\gamma$	
3	$z : \delta{\to}\beta$	
4	$v : \delta$	
5	...	
6	...	
7	...	
8	...	
9	$? : \gamma$	
10	$\lambda v : \delta.? : \delta{\to}\gamma$	λ-rule, 4, .
11	$\lambda z : \delta{\to}\beta.\lambda v : \delta.? : (\delta{\to}\beta){\to}\delta{\to}\gamma$	λ-rule, 3, .
12	$\lambda y : \alpha{\to}\beta{\to}\gamma.\lambda z : \delta{\to}\beta.\lambda v : \delta.? : (\alpha{\to}\beta{\to}\gamma){\to}(\delta{\to}\beta){\to}\delta{\to}\gamma$	λ-rule, 2, .
13	$\lambda x : \delta{\to}\delta{\to}\alpha.\lambda y : \alpha{\to}\beta{\to}\gamma.\lambda z : \delta{\to}\beta.\lambda v : \delta.? : \sigma$	λ-rule, 1, .

   The $? : \gamma$ should clearly be of the form $y ?_1 ?_2$ with $?_1 : \alpha$ and $?_2 : \beta$ ...and so forth. So one basically works "inside out" constructing the term. (This is basically "goal directed theorem proving".)

2. Find two terms of type $(\delta{\to}\delta{\to}\alpha){\to}(\gamma{\to}\alpha){\to}(\alpha{\to}\beta){\to}\delta{\to}\gamma{\to}\beta$

   ANSWER: (Construct the derivations yourself if you wish.)

   $\lambda x : \delta{\to}\delta{\to}\alpha.\lambda y : \gamma{\to}\alpha.\lambda z : \alpha{\to}\beta.\lambda v : \delta.\lambda w : \gamma.z\,(x\,v\,v)$

   $\lambda x : \delta{\to}\delta{\to}\alpha.\lambda y : \gamma{\to}\alpha.\lambda z : \alpha{\to}\beta.\lambda v : \delta.\lambda w : \gamma.z\,(y\,v)$

3. Find a term of type $((\alpha{\to}\beta){\to}\alpha){\to}(\alpha{\to}\alpha{\to}\beta){\to}\alpha$

   ANSWER: (Construct the derivation yourself if you wish.)

   $\lambda f : (\alpha{\to}\beta){\to}\alpha.\lambda g : \alpha{\to}\alpha{\to}\beta.f(\lambda x{:}\alpha.g\,x\,x)$

4. Find a term of type $((\alpha{\to}\beta){\to}\alpha){\to}(\alpha{\to}\alpha{\to}\beta){\to}\beta$

   ANSWER: We use the previous exercise! (Construct the derivation yourself if you wish.)

   $\lambda f : (\alpha{\to}\beta){\to}\alpha.\lambda g : \alpha{\to}\alpha{\to}\beta.g(f(\lambda x{:}\alpha.g\,x\,x))(f(\lambda x{:}\alpha.g\,x\,x))$.

On simple type theory à la Curry

1. Determine the most general unifiers of

   (a) $(\alpha{\to}\beta){\to}\gamma$ and $\alpha{\to}\beta{\to}\gamma$

   (b) $(\alpha{\to}\beta){\to}\gamma$ and $\gamma{\to}\alpha{\to}\beta$

   ANSWER: For clarity I first insert the brackets:

   (a) $(\alpha{\to}\beta){\to}\gamma \simeq \alpha{\to}(\beta{\to}\gamma)$, yielding $\alpha{\to}\beta \simeq \alpha$ and $\gamma \simeq \beta{\to}\gamma$. The first equation can't be fulfilled, so : "false".

   (b) $(\alpha{\to}\beta){\to}\gamma \simeq \gamma{\to}(\alpha{\to}\beta)$, yielding $\alpha{\to}\beta \simeq \gamma$ and $\gamma \simeq \alpha{\to}\beta$, yielding as a solution $\gamma := \alpha{\to}\beta$. This substitution is the most general unifier.

2. Compute the principal type of $\mathbf{S} := \lambda x.\lambda y.\lambda z.x\,z(y\,z)$.

   ANSWER: We assign type variables to all abstracted variables and applicative subterms:

   $\lambda x{:}\alpha.\lambda y{:}\beta.\lambda z{:}\gamma.((x\,z)^{\delta}(y\,z)^{\epsilon})^{\zeta}$.

   The type of this term is now $\alpha{\to}\beta{\to}\gamma{\to}\zeta$.

   To make this term typable, we have to solve the following equations.

$$\begin{array}{rcl}\alpha & \simeq & \gamma{\to}\delta\\ \beta & \simeq & \gamma{\to}\epsilon\\ \delta & \simeq & \epsilon{\to}\zeta.\end{array}$$

   Substituting the third gives

$$\begin{array}{rcl}\alpha & \simeq & \gamma{\to}\epsilon{\to}\zeta\\ \beta & \simeq & \gamma{\to}\epsilon\\ \delta & \simeq & \epsilon{\to}\zeta.\end{array}$$

   This has the shape of a substitution (only variables to the right; none of the variables at the right occurs at the left), so we have found the most general unifier.

   The principal type is now (substituting in the type $\alpha{\to}\beta{\to}\gamma{\to}\zeta$ above):

$$(\gamma{\to}\epsilon{\to}\zeta){\to}(\gamma{\to}\epsilon){\to}\gamma{\to}\zeta$$

3. Which of the following terms is typable? If it is, determine the *principal type*; if it isn't, show that the typing algorithm fails.

(a) $\lambda z\, x.z(x(\lambda y.y\, x))$

ANSWER: We assign type variables to all abstracted variables and applicative subterms:

$$\lambda z{:}\alpha.\lambda x{:}\beta.(z(x(\lambda y{:}\gamma.(y\, x)^{\delta}))^{\epsilon})^{\zeta}$$

The type of this term is now $\alpha{\to}\beta{\to}\zeta$.

To make this term typable, we have to solve the following equations.

$$\begin{array}{rcl} \gamma & \simeq & \beta{\to}\delta \\ \beta & \simeq & (\gamma{\to}\delta){\to}\epsilon \\ \alpha & \simeq & \epsilon{\to}\zeta \end{array}$$

Substituting the second, we get

$$\begin{array}{rcl} \gamma & \simeq & ((\gamma{\to}\delta){\to}\epsilon){\to}\delta \\ \beta & \simeq & (\gamma{\to}\delta){\to}\epsilon \\ \alpha & \simeq & \epsilon{\to}\zeta \end{array}$$

The first equation is not solvable, so: "false".

(b) $\lambda z\, x.z(x(\lambda y.y\, z))$

ANSWER: We assign type variables to all abstracted variables and applicative subterms:

$$\lambda z{:}\alpha.\lambda x{:}\beta.(z(x(\lambda y{:}\gamma.(y\, z)^{\delta}))^{\epsilon})^{\zeta}$$

The type of this term is now $\alpha{\to}\beta{\to}\zeta$.

To make this term typable, we have to solve the following equations.

$$\begin{array}{rcl} \gamma & \simeq & \alpha{\to}\delta \\ \beta & \simeq & (\gamma{\to}\delta){\to}\epsilon \\ \alpha & \simeq & \epsilon{\to}\zeta \end{array}$$

Substituting the first, we get

$$\begin{array}{rcl} \gamma & \simeq & \alpha{\to}\delta \\ \beta & \simeq & ((\alpha{\to}\delta){\to}\delta){\to}\epsilon \\ \alpha & \simeq & \epsilon{\to}\zeta \end{array}$$

and we have our most general unifier. So, the principal type is

$$(\epsilon{\to}\zeta){\to}(((\alpha{\to}\delta){\to}\delta){\to}\epsilon){\to}\zeta.$$

4. Compute the principal type of $M := \lambda x.\lambda y.x(y(\lambda z.x\, z\, z))(y(\lambda z.x\, z\, z))$.