Proving with Computer Assistance, 2IMF15

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Exercises on First order dependent type theory, λP, some answers

NB $\rightarrow$ binds strongest.

1. Give a precise derivation of the following judgment. (This is example 3 on the slides of the course.)

$$A : *,\ P : A \rightarrow *,\ a : A \vdash (P\,a) \rightarrow * : \Box$$

(Advise: give the derivation in "flag style", as it was shown in the lecture.)

ANSWER: We give it completely, using the $\rightarrow$-formation rule as a degenerate case of the Π-formation rule (if $x \notin \mathrm{FV}(B)$):

$$\frac{\Gamma \vdash A : * \quad \Gamma \vdash B : *}{\Gamma \vdash A \rightarrow B : *}\ \rightarrow\text{-form} \qquad \frac{\Gamma \vdash A : * \quad \Gamma, x{:}A \vdash B : *}{\Gamma \vdash \Pi x{:}A.B : *}\ \Pi\text{-form}$$

$$\begin{array}{r|l|l}
1 & * : \Box & \\
2 & \quad A : * & \text{var, } 1 \\
3 & \quad A \rightarrow * : \Box & \rightarrow\text{-form, } 2,\ 1 \\
4 & \quad\quad P : A \rightarrow * & \text{var, } 3 \\
5 & \quad\quad\quad a : A & \text{var, } 2 \\
6 & \quad\quad\quad P\,a : * & \text{app, } 4,\ 5 \\
7 & \quad\quad\quad P\,a \rightarrow * : \Box & \rightarrow\text{-form, } 6,\ 1
\end{array}$$

2. Find a term of the following type and write down the context in which this term is typed. (This is example 5 on the slides of the course.)

$$(\Pi x{:}A.P\,x \rightarrow Q\,x) \rightarrow (\Pi x{:}A.P\,x) \rightarrow \Pi x{:}A.Q\,x$$

Do this by giving a derivation in "flag style", where you may omit derivations of the well-formedness of types.

ANSWER: Write σ for $(\Pi x{:}A.P\,x \rightarrow Q\,x) \rightarrow (\Pi x{:}A.P\,x) \rightarrow \Pi x{:}A.Q\,x$.

1	$A : *$	
2	$P : A \to *$	
3	$Q : A \to *$	
4	$h : \Pi x{:}A.P\,x {\to} Q\,x$	
5	$g : \Pi x{:}A.P\,x$	
6	$x : A$	
7	$h\,x : P\,x \to Q\,x$	app, 4, 6
8	$g\,x : P\,x$	app, 5, 6
9	$h\,x(g\,x) : Q\,x$	app, 7, 8
10	$\lambda x{:}A.h\,x(g\,x) : \Pi x{:}A.Q\,x$	λ-rule, 6, 9
11	$\lambda g{:}\Pi x{:}A.P\,x.\lambda x{:}A.h\,x(g\,x) : (\Pi x{:}A.P\,x) \to \Pi x{:}A.Q\,x$	λ-rule, 5, 10
12	$\lambda h{:}\Pi x{:}A.P\,x{\to}Q\,x.\lambda g{:}\Pi x{:}A.P\,x.\lambda x{:}A.h\,x(g\,x) : \sigma$	λ-rule, 4, 11

So:

$$A : *, P : A \to *, Q : A \to * \vdash \lambda h{:}\Pi x{:}A.P\,x{\to}Q\,x.\lambda g{:}\Pi x{:}A.P\,x.\lambda x{:}A.h\,x(g\,x) : \sigma$$

3. Find a term of the following type and write down the context in which this term is typed.

$$(\Pi x{:}A.P\,x{\to}\Pi z{:}A.R\,x\,z){\to}(\Pi x{:}A.P\,x){\to}\Pi z{:}A.R\,z\,z).$$

(**NB**. Read this type in the proper way: $\to$ binds stronger than Π!)

ANSWER: write τ for $(\Pi x{:}A.P\,x{\to}\Pi z{:}A.R\,x\,z){\to}(\Pi x{:}A.P\,x){\to}\Pi z{:}A.R\,z\,z)$

$$A : *, P : A \to *, R : A \to A \to * \vdash$$
$$\lambda h : \Pi x{:}A.P\,x{\to}\Pi z{:}A.R\,x\,z.\lambda g : \Pi x{:}A.P\,x.\lambda y : A.h\,y(g\,y)\,y : \tau$$

4. Give a term M of type $\Pi x{:}A.P(f(f\,x))$ in the context

$$\begin{aligned}\Gamma \quad := \quad & A : *, P : A{\to}*, f : A{\to}A, g : A{\to}A,\\ & h : \Pi x{:}A.P(f\,x){\to}P(g\,x), k : \Pi x, y{:}A.(P\,x{\to}P\,y){\to}P(f\,x).\end{aligned}$$

Also give a derivation of $\Gamma \vdash M : \Pi x{:}A.P(f(f\,x))$ in 'short form', so you don't have to show he well-formedness of the types.

ANSWER (only the term):

$$\lambda x : A.k(f\,x)(g\,x)(h\,x)$$

5. Find a term of the following type and write down the context in which this term is typed.

$$(\Pi x{:}A.P\,x{\to}Q){\to}(\Pi x{:}A.P\,x){\to}Q$$

What is special about your context? It should somehow explicitly state that the type A is not empty. How? Why?

ANSWER:

$A : *, P : A \to *, Q : *, \mathbf{a} : \mathbf{A} \vdash$
$\lambda h{:}\Pi x{:}A.P\,x{\to}Q.\lambda g{:}\Pi x{:}A.P\,x.h\,a\,(g\,a) : (\Pi x{:}A.P\,x{\to}Q){\to}(\Pi x{:}A.P\,x){\to}Q$

We need a declaration of a variable $a : A$ in teh context, stating that A is not empty. If we don't have a term of type A, we can not construct a term of this type, so if A is just a variable in the context, teh only thing we can do is to decalre $a : A$ as well.

Note that, if A is the "empty type", the type $(\Pi x{:}A.P\,x{\to}Q){\to}(\Pi x{:}A.P\,x){\to}Q$, interpreted as a formula states something that is just not true:
$\forall x{:}A.P\,x{\to}Q$ and $\forall x{:}A.P\,x$ are both vacuouslu try of A is empty, but Q need not be.

6. Find a term from the given hypotheses of the following type and write down the context in which this term is typed.

$$\begin{aligned}&\forall x.\,(P(x) \to R(x, f(x))),\\ &\forall x, y.\,(R(x, y) \to R(y, x)),\\ &\forall x, y.\,(R(x, y) \to R(f(y), x)) \quad \vdash \quad \forall x.\,(P(x) \to R(f(x), f(x)))\end{aligned}$$

ANSWER: We only give the term, type and context. The derivation you have to add yourself!

In context $D : *, f : D \to D, P : D \to *, R : D \to D \to *$,
$t : \Pi x{:}D.P\,x \to R\,x\,(f\,x)$,
$s : \Pi x, y{:}D.R\,x\,y \to R\,y\,x$,
$q : \Pi x, y{:}D.R\,x\,y \to R\,(f\,y)\,x$, we have

$$\lambda x{:}D.\lambda h{:}P\,x.\,q\,(f\,x)\,x\,(s\,x\,(f\,x)\,(t\,x\,h)) : \Pi x{:}D.P\,x \to R\,(f\,x)\,(f\,x).$$