Proving with Computer Assistance, 2IF65
Herman Geuvers, TUE
Some answers for the exercises on Lecture 5

NB → binds strongest.

1. Give a precise derivation of the following judgment. (This is example 3 on the slides of the course.)

\[ A : *, P : A \to *, a : A \vdash (Pa) \to * : \Box \]

(Advise: give the derivation in “flag style”, as it was shown in the lecture.)

ANSWER: We give it completely, using the \( \to \)-formation rule as a degenerate case of the \( \Pi \)-formation rule (if \( x \notin \text{FV}(B) \)):

\[
\frac{
\Gamma \vdash A : * \\
\Gamma \vdash B : *
}{
\Gamma \vdash A \to B : *
}\quad \rightarrow \text{-form}
\]

\[
\frac{
\Gamma \vdash A : * \\
\Gamma, x : A \vdash B : *
}{
\Gamma \vdash \Pi x : A.B : *
}\quad \Pi \text{-form}
\]

\[
\begin{array}{|c|}
\hline
1 & * : \Box \\
2 & A : * \quad \text{var, 1} \\
3 & A \to * : \Box \quad \rightarrow \text{-form, 2, 1} \\
4 & P : A \to * \quad \text{var, 3} \\
5 & a : A \quad \text{var, 2} \\
6 & Pa : * \quad \text{app, 4, 5} \\
7 & Pa \to * : \Box \quad \rightarrow \text{-form, 6, 1} \\
\hline
\end{array}
\]

2. Find a term of the following type and write down the context in which this term is typed. (This is example 5 on the slides of the course.)

\[(\Pi x : A.P) x \to Q x) \to (\Pi x : A.P) x \to (\Pi x : A.Q) x\]

Do this by giving a derivation in “flag style”, where you may omit derivations of the well-formedness of types.

ANSWER: Write \( \sigma \) for \((\Pi x : A.P) x \to Q x) \to (\Pi x : A.P) x \to (\Pi x : A.Q) x\).
1. $A : *$
2. $P : A \to *$
3. $Q : A \to *$
4. $h : \Pi x : A. Px \to Q x$
5. $g : \Pi x : A. Px$
6. $x : A$
7. $hx : Px \to Q x$
8. $gx : P x$
9. $h x(g x) : Q x$
10. $\lambda x : A. h x(g x) : \Pi x : A.Q x$
11. $\lambda g : \Pi x : A. P x. \lambda x : A. h x(g x) : \sigma$
12. $\lambda h : \Pi x : A. P x \to Q x. \lambda g : \Pi x : A. P x. \lambda x : A. h x(g x) : \sigma$

So:

$$A : *, P : A \to *, Q : A \to * \vdash \lambda h : \Pi x : A. P x \to Q x. \lambda g : \Pi x : A. P x. \lambda x : A. h x(g x) : \sigma$$

3. Find a term of the following type and write down the context in which this term is typed.

$$(\Pi x : A. P x \to \Pi z : A.R x z) \to (\Pi x : A. P x) \to \Pi z : A.R z z).$$

(\textbf{NB.} Read this type in the proper way: $\to$ binds stronger than $\Pi$!)

\textbf{ANSWER:} write $\tau$ for $$(\Pi x : A. P x \to \Pi z : A.R x z) \to (\Pi x : A. P x) \to \Pi z : A.R z z)$$

$$A : *, P : A \to *, R : A \to A \to * \vdash \lambda h : \Pi x : A. P x \to \Pi z : A.R x z. \lambda g : \Pi x : A. P x. \lambda y : A. h y g y : \tau$$

4. Give a term $M$ of type $\Pi x : A. P(f(f x))$ in the context

$$\Gamma := A : *, P : A \to *, f : A \to A, g : A \to A,$$

$$h : \Pi x : A. P(f x) \to P(g x), k : \Pi x, y : A. (P x \to P y) \to P(f x).$$

Also give a derivation of $\Gamma \vdash M : \Pi x : A. P(f(f x))$ in ‘short form’, so you don’t have to show he well-formedness of the types.

\textbf{ANSWER (only the term)}:

$$\lambda x : A. k(f x)(g x)(h x)$$
5. Find a term of the following type and write down the context in which this term is typed.

\[(\Pi x.A.P \rightarrow Q) \rightarrow (\Pi x.A.P \rightarrow Q)\]

What is special about your context? It should somehow explicitly state that the type \(A\) is not empty. How? Why?

**ANSWER:**

\[A : *, P : A \rightarrow *, Q : *, a : A \vdash\]

\[\lambda h : \Pi x.A.P \rightarrow Q. \lambda g : \Pi x.A.P . x. h a (g a) : (\Pi x.A.P \rightarrow Q) \rightarrow (\Pi x.A.P \rightarrow Q)\]

We need a declaration of a variable \(a : A\) in the context, stating that \(A\) is not empty. If we don’t have a term of type \(A\), we can not construct a term of this type, so if \(A\) is just a variable in the context, the only thing we can do is to declare \(a : A\) as well.

Note that, if \(A\) is the “empty type”, the type \((\Pi x.A.P \rightarrow Q) \rightarrow (\Pi x.A.P \rightarrow Q)\), interpreted as a formula states something that is just not true: \(\forall x.A.P \rightarrow Q\) and \(\forall x.A.P \rightarrow Q\) are both vacuously true if \(A\) is empty, but \(Q\) need not be.