For substitution (substitute $N$ for $x$ in $M$) one sometimes writes $M[x := N]$ (e.g. Takahashi) and sometimes $M[N/x]$ (the exercise sheet).

## Exercise Church-Rosser

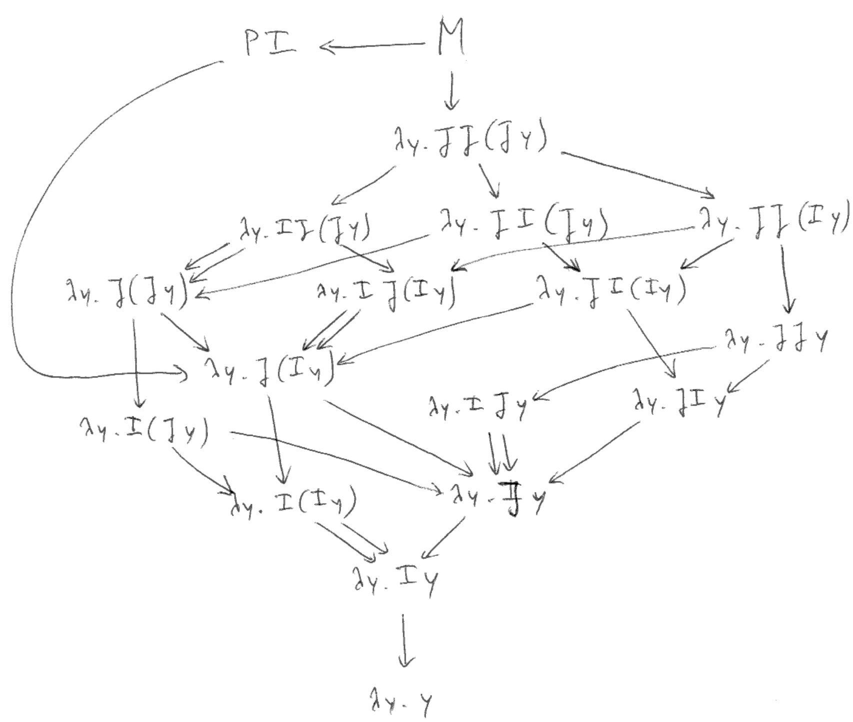
1

1. $M = (\lambda xy.\, xx(xy))(II)$

   $P = \lambda xy.\, xx(xy)$

   $J = II$

(a) We omit duplicates in the graph.
Note that $\lambda y.\, II(xI)$ is just $\lambda y.\, J(xI)$ etc.




(b) $M^* = \lambda y.\, II(Iy)$

$(M^*)^* = \lambda y.\, Iy$

1c



$$\boxed{M \Rightarrow M^*}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{x \Rightarrow x \quad x \Rightarrow x}{xx \Rightarrow xx} \qquad \cfrac{x \Rightarrow x \quad y \Rightarrow y}{xy \Rightarrow xy}}{xx(xy) \Rightarrow xx(xy)}}{\lambda y.xx(xy) \Rightarrow \lambda y.xx(xy)} \qquad \cfrac{z \Rightarrow z \quad \lambda z.z \Rightarrow \lambda z.z}{(\lambda z.z)I \Rightarrow I}}{(\lambda xy.xx(xy))(II) \Rightarrow \lambda y.II(Iy)}$$

$$\boxed{I = \lambda z.z} \qquad \cfrac{\cfrac{\cfrac{z \Rightarrow z \quad \cfrac{z \Rightarrow z}{I \Rightarrow I}}{II \Rightarrow I} \qquad \cfrac{z \Rightarrow z \quad y \Rightarrow y}{Iy \Rightarrow y}}{II(Iy) \Rightarrow Iy}}{\lambda y.II(Iy) \Rightarrow \lambda y.Iy} \qquad \boxed{I = \lambda z.z}$$

$$\boxed{M^* \Rightarrow (M^*)^*}$$

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2 (a) Change the cases $(\beta_3^*)$ and $(\beta_4^*)$ to the following

$(\beta_5^*)$ If $M_1^* \equiv \lambda x.P$, then $(M_1 M_2)^* \equiv P[x := M_2^*]$

(b)
$$\frac{M \Rightarrow \lambda x.P \qquad N \Rightarrow N'}{MN \Rightarrow \cancel{P[N'/x]}\ P[x := N']}$$

<u>IH</u> $\lambda x.P \Rightarrow M^*$, $N' \Rightarrow N^*$

<u>To prove</u>: $\cancel{P[N'/x] \Rightarrow (MN)^*}$ $P[x := N'] \Rightarrow (MN)^*$

From $\lambda x.P \Rightarrow M^*$ we deduce that $M^* = \lambda x.Q$ with $P \Rightarrow Q$

So $(MN)^* = Q[x := N^*]$.

we have $P \Rightarrow Q$ and $N' \Rightarrow N^*$, so by substitution (property (3) in the Takahashi paper) we conclude $P[x := N'] \Rightarrow Q[x := N^*] = (MN)^*$ ☒

(3) (a) (3)

$$\dfrac{\dfrac{y \Rightarrow y}{\lambda x. y x \Rightarrow y} \qquad I \Rightarrow I}{(\lambda y. \lambda x. y\, x)\, I \Rightarrow \underset{y[y:=I]}{\overset{\|}{I}}}$$

We don't have $(\lambda y. \lambda x. y x)\, I \Rightarrow I$

because if this were derivable, a derivation has to have the following shape

$$\dfrac{\dfrac{\circledast}{\lambda x. y x \Rightarrow M'} \qquad I \Rightarrow I}{(\lambda y. \lambda x\, y\, x)\, I \Longrightarrow M'[y := I] \equiv I}$$

Note $M'[y = I] \equiv I$ can be because of

(i) $M' \equiv y$

(ii) $M' \equiv I$

In case (i) we must have a derivation $\dfrac{\circledast}{\lambda x. y x \Rightarrow y}$

but that can't be, because a $\lambda$-abstraction only parallel reduces to another $\lambda$-abstraction

In case (ii) we must have $\dfrac{\circledast}{\lambda x\, y x \Rightarrow \lambda x. x}$

and so $y x \Rightarrow x$, which is not the case either.

So: we can't have such a derivation $\boxtimes$

(b) $(\lambda x. M)^* = \begin{cases} P^* & \text{if } M = Px \text{ with } x \notin FV(P) \\ \lambda x. M^* & \text{otherwise.} \end{cases}$