Proving with Computer Assistance

Exercises on Normalization

See the Slides and the Course Notes by Herman Geuvers for the definitions.

1. In the proof of WN for $\lambda\!\to$, the height of a type $h(\sigma)$ is defined by

   - $h(\alpha) := 0$
   - $h(\sigma_1 \to \ldots \to \sigma_n \to \alpha) := \max(h(\sigma_1), \ldots, h(\sigma_n)) + 1.$

   Prove that this is the same as taking as the second clause

   - $h(\sigma \to \tau) := \max(h(\sigma) + 1, h(\tau)).$

2. In the proof of WN for $\lambda\!\to$, it is stated that, if $M \longrightarrow_\beta N$ by contracting a redex of maximum height, $h(M)$, that does not contain any other redex of height $h(M)$, then this does not create a new redex of maximum height.

   Show that this holds for the case

   $$\begin{array}{rcl} M & = & (\lambda x : A.x\,(\lambda v : B.x\,\mathbf{I}))(\lambda z : C.z\,(\mathbf{I}\,\mathbf{I})) \\ & \longrightarrow_\beta & (\lambda z : C.z\,(\mathbf{I}\,\mathbf{I}))(\lambda v : B.(\lambda z : C.z\,(\mathbf{I}\,\mathbf{I}))\,\mathbf{I}) \quad = \quad N \end{array}$$

   where $B = \alpha \to \alpha$, $C = B \to B$ and $A = C \to B$.

   Also show that $m(M) >_l m(N)$.

3. Suppose X, Y, and Z are properties of λ-terms. Then we can have the following situations: If M satisfies property X and N satisfies property Y, then

   (a) Yes, property Z always holds (so $\forall M, N(M \in X \wedge N \in Y \Rightarrow M\,N \in Z)$

   (b) No, property Z never holds (so $\forall M, N(M \in X \wedge N \in Y \Rightarrow M\,N \notin Z)$

   (c) Undec, property Z holds for some M, N, and doesn't hold for some other M, N (so $\exists M, N(M \in X \wedge N \in Y \wedge M\,N \in Z$ and $\exists M, N(M \in X \wedge N \in Y \wedge M\,N \notin Z)$

   Fill in the following diagram with Yes, No and Undec and motivate your answers. In case of Undec, give M, N for both cases.

	$N \in$ WN	$N \in \neg$SN
$M \in$ WN	$M\,N \in$ WN?	$M\,N \in$ SN?
$M \in$ SN	$M\,N \in$ SN?	$M\,N \in \neg$SN?