

Proving with Computer Assistance

Exercises on Normalization

See the Slides and the Course Notes by Herman Geuvers for the definitions.

1. In the proof of WN for $\lambda \rightarrow$, the height of a type $h(\sigma)$ is defined by

- $h(\alpha) := 0$
- $h(\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \alpha) := \max(h(\sigma_1), \dots, h(\sigma_n)) + 1$.

Prove that this is the same as taking as the second clause

- $h(\sigma \rightarrow \tau) := \max(h(\sigma) + 1, h(\tau))$.

2. Consider the following term $N : A$, where $A = \alpha \rightarrow \alpha$ and $\mathbf{I}_1 : A$ and $\mathbf{I}_2 : A \rightarrow A$ and $\mathbf{I}_3 : A$ and $\mathbf{I}_4 : A$ are copies of the well-known λ -term \mathbf{I} ($:= \lambda x.x$).

$$N \quad := \quad \lambda y:\alpha.(\lambda x:A \rightarrow A.\mathbf{I}_1(x \mathbf{I}_4(\mathbf{I}_3 y))) \mathbf{I}_2$$

- (a) Determine $m(N)$, the *measure* of N as defined in the weak normalization proof.
 - (b) Determine which redex will be contracted following the strategy in the weak normalization proof, obtaining a term N' .
 - (c) Determine $m(N')$, the measure of this reduct of N .
3. In the proof of WN for $\lambda \rightarrow$, it is stated that, if $M \rightarrow_\beta N$ by contracting a redex of maximum height, $h(M)$, that does not contain any other redex of height $h(M)$, then this does not create a new redex of maximum height.

Show that this holds for the case

$$\begin{aligned} M &= (\lambda x : A.x(\lambda v : B.x \mathbf{I}))(\lambda z : C.z(\mathbf{II})) \\ &\rightarrow_\beta (\lambda z : C.z(\mathbf{II}))(\lambda v : B.(\lambda z : C.z(\mathbf{II})) \mathbf{I}) = P \end{aligned}$$

where $B = \alpha \rightarrow \alpha$, $C = B \rightarrow B$ and $A = C \rightarrow B$.

Also show that $m(M) >_l m(P)$.

4. Suppose X , Y , and Z are properties of λ -terms. Then we can have the following situations: If M satisfies property X and N satisfies property Y , then
 - (a) **Yes**, property Z always holds (so $\forall M, N (M \in X \wedge N \in Y \Rightarrow M N \in Z)$)
 - (b) **No**, property Z never holds (so $\forall M, N (M \in X \wedge N \in Y \Rightarrow M N \notin Z)$)
 - (c) **Undec**, property Z holds for some M, N , and doesn't hold for some other M, N (so $\exists M, N (M \in X \wedge N \in Y \wedge M N \in Z$ and $\exists M, N (M \in X \wedge N \in Y \wedge M N \notin Z)$)

Fill in the following diagram with **Yes**, **No** and **Undec** and motivate your answers. In case of **Undec**, give M, N for both cases.

| | | |
|-------------------|----------------------|--------------------------|
| | $N \in \text{WN}$ | $N \in \neg\text{SN}$ |
| $M \in \text{WN}$ | $M N \in \text{WN}?$ | $M N \in \text{SN}?$ |
| $M \in \text{SN}$ | $M N \in \text{SN}?$ | $M N \in \neg\text{SN}?$ |

5. Prove that *type reduction* is SN for λ_2 a la Church. (Define a simple measure on terms that decreases with type reduction.)