Proving with Computer Assistance
Lecture 10

Higher Order Logic and the Calculus of Constructions

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The Barendregt cube

Barendregt cube: 8 typed λ-calculi, defined in one coherent way.
Generalization: Berardi & Terlouw: Pure Type Systems

framework for defining and studying typed λ-calculi
PTS = pure type system

the PTS rules are basically the λP rules as presented before.

variations on the product rule

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A.\, B : s_2}$$

$$\begin{array}{ll}
\lambda P & s_1 = *,\ s_2 \in \{*, \Box\} \\
 & \\
 & (s_1, s_2) \in \{(*, *), (*, \Box)\} \\
\lambda{\to} & (s_1, s_2) \in \{(*, *)\} \\
\lambda 2 & (s_1, s_2) \in \{(*, *), (\Box, *)\} \\
\lambda C & (s_1, s_2) \in \{(*, *), (*, \Box), (\Box, *), (\Box, \Box)\}
\end{array}$$

(axiom) $\vdash * : \square$

(var) $$\frac{\Gamma \vdash A : s}{\Gamma, x{:}A \vdash x : A}$$ (weak) $$\frac{\Gamma \vdash A : s \quad \Gamma \vdash M : C}{\Gamma, x{:}A \vdash M : C}$$

(Π) $$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \Pi x{:}A.B : s_2} \text{ if } (s_1, s_2) \in \mathcal{R}$$

(λ) $$\frac{\Gamma, x{:}A \vdash M : B \quad \Gamma \vdash \Pi x{:}A.B : s}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B}$$

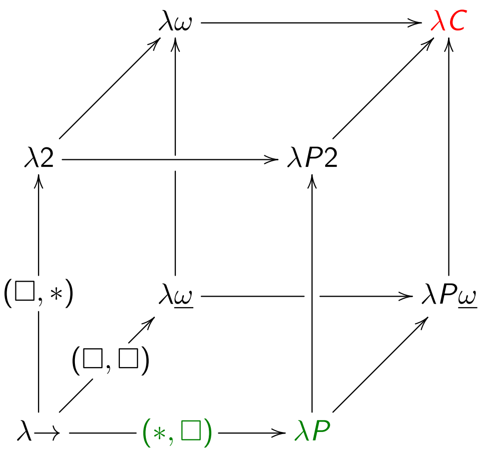
(app) $$\frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B[N/x]}$$

(conv) $$\frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma \vdash M : B} \text{ if } A =_\beta B$$

$$(\Pi) \quad \frac{\Gamma \vdash A : s_1 \qquad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \Pi x{:}A.B : s_2} \quad \text{if } (s_1, s_2) \in \mathcal{R}$$

System	$\mathcal{R}$			
$\lambda\!\to$	$(*, *)$			
$\lambda 2$ (system F)	$(*, *)$	$(\Box, *)$		
λP (LF)	$(*, *)$		$(*, \Box)$	
$\lambda\overline{\omega}$	$(*, *)$			$(\Box, \Box)$
λP2	$(*, *)$	$(\Box, *)$	$(*, \Box)$	
$\lambda\omega$ (system Fω)	$(*, *)$	$(\Box, *)$		$(\Box, \Box)$
$\lambda P\overline{\omega}$	$(*, *)$		$(*, \Box)$	$(\Box, \Box)$
λPω (CC)	$(*, *)$	$(\Box, *)$	$(*, \Box)$	$(\Box, \Box)$

the Barendregt cube



Calculus of Constructions

$\lambda{\to}$ in this presentation is equivalent to $\lambda{\to}$ as presented before. Similarly for $\lambda 2$, λP, ... This cube also gives a fine structure for the

Calculus of Constructions, *CC* *(Coquand and Huet)*

- Polymorphic data types on the $*$-level, e.g. $\Pi\alpha{:}*.\alpha{\to}(\alpha{\to}\alpha){\to}\alpha : *$.
- Predicate domains on the $\square$-level, e.g. $N{\to}N{\to}* : \square$
- Logic on the $*$-level, e.g. $\varphi \wedge \psi := \Pi\alpha{:}*.(\varphi{\to}\psi{\to}\alpha){\to}\alpha : *$.
- Universal quantification (first and higher order), e.g. $\Pi P{:}N{\to}*.\Pi x{:}N.Px{\to}Px : *$.

Examples

- Induction

$$\forall P{:}N{\to}* \left(\ (P\,0) \to (\forall x{:}N.(P\,x \to P(S\,x))) \to \forall x{:}N.P\,x \ \right)$$

- Higher order predicates/functions: transitive closure of a relation R

$$\begin{aligned}&\lambda R : A{\to}A{\to}*\,.\,\lambda x, y : A.\\ &\qquad (\forall Q : A{\to}A{\to}*\,.\,(\mathrm{trans}(Q) \to (R \subseteq Q) \to Q\,x\,y))\end{aligned}$$

  of type

$$(A{\to}A{\to}*){\to}(A{\to}A{\to}*)$$

Example trans clos higher order and inductively

- transitive closure in higher order logic:

$$\lambda R : A{\to}A{\to} * . \lambda x, y : A.$$
$$(\forall Q : A{\to}A{\to} * . (\mathsf{trans}(Q) \to (R \subseteq Q) \to Q\,x\,y))$$

of type

$$(A{\to}A{\to}*){\to}(A{\to}A{\to}*)$$

- transitive closure inductively:

```
Inductive TrclosInd (R : A->A->Prop) : A -> A -> Prop :=
| sub : forall x y : A, R x y -> TrclosInd x y
| trans : forall x y z : A,
        TrclosInd x y -> TrclosInd y z -> TrclosInd x z.
```

Exercise trans clos higher order

Given the transitive closure of a binary relation, defined in higher order logic:

$$\begin{aligned}\mathsf{trclos}\, R \quad &:= \quad \lambda x, y{:}A. \\ &\qquad (\forall Q{:}A{\to}A{\to}*.(\mathsf{trans}(Q){\to}(R \subseteq Q){\to}(Q\,x\,y))).\end{aligned}$$

1. Prove that the transitive closure is transitive.
2. Prove that the transitive closure of R contains R.

Higher order logic HOL

In higher order logic (originally due to Church[1940]) we have:

- higher order domains: D, $D \to \text{Prop}$, $(D \to \text{Prop}) \to \text{Prop}$, etc (sets of predicates over predicates over ...).
- higher order function domains: $(D \to D) \to D$, $((D \to D) \to D) \to D$, etc.
- $\forall$-quantification over all domains

We can do Higher Order Logic in Coq
In Coq we often have the choice to define sets/predicates/relations inductively or via higher order logic. The Standard Library uses inductive representations.

Definability of other connectives (constructively)

$$\begin{array}{rcl} \bot & := & \forall \alpha{:}\, * . \alpha \\ \varphi \wedge \psi & := & \forall \alpha{:}\, * . (\varphi \to \psi \to \alpha) \to \alpha \\ \varphi \vee \psi & := & \forall \alpha{:}\, * . (\varphi \to \alpha) \to (\psi \to \alpha) \to \alpha \\ \exists x{:}\sigma . \varphi & := & \forall \alpha{:}\, * . (\forall x{:}\sigma . \varphi \to \alpha) \to \alpha \end{array}$$

Idea:
The definition of a connective is an encoding of the elimination rule.

Existential quantifier

$$\exists x{:}\sigma.\varphi := \forall \alpha{:}*.(\forall x{:}\sigma.\varphi \to \alpha) \to \alpha$$

Derivation of the elimination rule in HOL.

$$\frac{\exists x{:}\sigma.\varphi \qquad \begin{array}{c}[\varphi] \\ \vdots \\ C\end{array}}{C}\; x \notin \mathrm{FV}(C, \text{ass.}) \qquad\qquad \dfrac{\dfrac{\exists x{:}\sigma.\varphi}{(\forall x{:}\sigma.\varphi \to C) \to C} \qquad \dfrac{\begin{array}{c}[\varphi] \\ \vdots \\ C\end{array}}{\forall x{:}\sigma.\varphi \to C}}{C}$$

Equality

Equality is definable in higher order logic:

> *t and q terms are equal if they share the same properties (Leibniz equality)*

Definition in HOL (for $t, q : A$):

$$t =_A q := \forall P{:}A\to *.(Pt \to Pq)$$

- This equality is reflexive and transitive (easy)
- It is also symmetric(!) Trick: find a "smart" predicate P

Exercise: Prove reflexivity, transitivity and symmetry of $=_A$.

CC versus HOL

Question: is the type theory CC really isomorphic with HOL?
No: only if we disambiguate $*$ into Set and Prop (or $*_s$ and $*_p$).
This is the type theory of Coq.

Properties of CC

- Uniqueness of types
  If $\Gamma \vdash M : A$ and $\Gamma \vdash M : B$, then $A =_\beta B$.
- Subject Reduction
  If $\Gamma \vdash M : A$ and $M \to_\beta N$, then $\Gamma \vdash N : A$.
- Strong Normalization
  If $\Gamma \vdash M : A$, then all β-reductions from M terminate.

Proof of SN is a really difficult.

Decidability Questions

$$\begin{array}{ll} \Gamma \vdash M : \sigma? & \text{TCP} \\ \Gamma \vdash M : ? & \text{TSP} \\ \Gamma \vdash ? : \sigma & \text{TIP} \end{array}$$

For CC:

- TIP is undecidable
- TCP/TSP: simultaneously.
  The type checking algorithm is close to the one for λP. (In λP we had a judgement of correct context; this form of judgement could also be introduced for CC)