Eindhoven University of Technology

Faculty of Mathematics and Computer Science

Written Exam **Proving with Computer Assistance 2IMF15**, Tuesday April 6, 2021, 09.00–12.00 **All paper material: lecture notes, own notes etc. can be consulted during the exam.** The maximum number of points per exercise is indicated in the margin. Maxium 100 points in total.
NB Typing derivations may be given in "flag style" or in "sequent style".

---

1. Consider the following term "with holes" P, where $\mathbf{I}_1$ and $\mathbf{I}_2$ are copies of the well-known λ-term $\lambda z.z$.

$$(\lambda f :?_1.\lambda x :?_2.\mathbf{I}_1(f\,x))\,(\lambda g{:}\alpha \to \beta.\lambda y{:}\alpha.g\,(\mathbf{I}_2\,y))$$

(10) (a) Fill in types for the ? in P, give the types for $\mathbf{I}_1$ and $\mathbf{I}_2$ and give the type of P itself in simple type theory ($\lambda\!\!\to$) à la Church. Use $A := \alpha \to \beta$ as abbreviation. NB. You don't have to give a typing derivation.

(5) (b) Compute $m(P)$, where $m(-)$ is the measure of a term, as used in the proof of weak normalization (WN) for $\lambda\!\!\to$. Indicate which redex will be contracted according to the strategy described in the WN proof.

(10) 2. Give a term M in simple type theory ($\lambda\!\!\to$) à la Church with type

$$(\alpha \to \beta \to \gamma) \to ((\alpha \to \gamma) \to \alpha) \to \beta \to \gamma.$$

Give a typing derivation that gives the type of M.

3. As usual, we define $\mathbf{S}$ as $\lambda x\,y\,z.x\,z\,(y\,z)$, and $\mathbf{I}$ as $\lambda p.p$.

(7) (a) Give the reduction graph of $Q := \mathbf{S}\,\mathbf{I}\,\mathbf{I}$.

(5) (b) Compute Q^*, and $(Q^*)^*$ where $(-)^*$ is the "complete development" function used in the Church-Rosser proof.

(16) 4. For each of the following two terms, compute its principle type in simple type theory ($\lambda\!\!\to$) à la Curry, if it exists. Give the end result and show your computation.

- $\lambda x\,y.y\,(y\,x\,x)$
- $\lambda x\,y.y\,x\,(y\,x)$

(10) 5. Let $x : \top$, where $\top := \forall\alpha : *.\alpha \to \alpha$. Give a type to the following term in $\lambda 2$ à la Curry. Also give the typing derivation of your result.

$$\lambda y.x\,(y\,(x\,(\lambda z.z\,z)))$$

Continue on the other side

6. In $\lambda 2$ à la Church we define two types of binary trees: $\mathbb{ET}(C)$, with empty leaves and nodes labelled with terms in C, and $\mathbb{LT}(A, B)$, with leaves in A and nodes labelled with terms in B:

$$\begin{array}{rcl} \mathbb{ET}(C) & := & \forall \alpha{:}*.\alpha \to (C \to \alpha \to \alpha \to \alpha) \to \alpha \\ \mathbb{LT}(A,B) & := & \forall \alpha{:}*.(A \to \alpha) \to (B \to \alpha \to \alpha \to \alpha) \to \alpha. \end{array}$$

Each of these data types has 2 constructors: $\mathsf{eleaf} : \mathbb{ET}(C)$ and $\mathsf{ejoin} : C \to \mathbb{ET}(C) \to \mathbb{ET}(C) \to \mathbb{ET}(C)$ and $\mathsf{lleaf} : A \to \mathbb{LT}(A,B)$ and $\mathsf{ljoin} : B \to \mathbb{LT}(A,B) \to \mathbb{LT}(A,B) \to \mathbb{LT}(A,B)$.

(7) (a) Given the types A, B, define the function $F : \mathbb{LT}(A,B) \to \mathbb{ET}(B)$ that replaces all leaves by an empty leaf.

(7) (b) Given the types A, B, C, and the term $h : C \to B$ and $a : A$, define the function $G : \mathbb{ET}(C) \to \mathbb{LT}(A,B)$ that puts a in all of the leaves and applies h to all the nodes.

(8) 7. In the system λP, give a term of type $\Pi x, y{:}A.R\,x\,y \to R\,y\,x$ in the context

$$\begin{array}{rcl} \Gamma & := & A : *, R : A \to A \to *, \\ & & p : \Pi x, y, z{:}A.R\,x\,y \to R\,x\,z \to R\,y\,z, \\ & & k : \Pi x{:}A.R\,x\,x. \end{array}$$

Give a typing derivation in short form (where you don't have to establish the well-formedness of the types themselves).

8. In the Calculus of Constructions λC, given $A : *, R : A \to A \to *, Q : A \to A \to *$, we define, $\overline{R;Q}$ (the *composition of* R *and* Q) as follows

$$\overline{R;Q} := \lambda x, y{:}A.\Pi \alpha{:}*.(\Pi z{:}A.R\,x\,z \to Q\,z\,y \to \alpha) \to \alpha.$$

(8) (a) Show that $\overline{R;Q}$ is indeed the composition of R and Q, that is: give a term of type

$$\Pi a, b, c{:}A.R\,a\,b \to Q\,b\,c \to \overline{R;Q}\,a\,c.$$

(7) (b) Prove that, if R is transitive, then then $\overline{R;R} \subseteq R$. That is, give a term of type

$$\mathsf{trans}\,R \to \Pi x, y{:}A.\,\overline{R;R}\,x\,y \to R\,x\,y,$$

where, $\mathsf{trans}\,R := \Pi x, y, z{:}A.R\,x\,y \to R\,y\,z \to R\,x\,z.$

NB. In both cases, give a typing derivation in short form.

END