Eindhoven University of Technology

Faculty of Mathematics and Computer Science

Written Exam **Proving with Computer Assistance 2IF65**

Wednesday July 2 2010, 14.00–17.00

The maximum number of points per exercise is indicated in the margin. (Maxium 100 points in total.) **NB** Typing derivations may be given in “flag style” or in “sequent style”.

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(10) 1. Show that the following term P is typable in simple type theory ($\lambda{\to}$) à la Church.

$$\lambda x : (\alpha \to \beta) \to \beta \to \alpha . \lambda y : \beta . x\,(\lambda z : \alpha . y)\,y.$$

Give a typing derivation that gives the type of P.

(10) 2. Construct a term M of type $((((\alpha \to \alpha) \to \alpha) \to \alpha) \to \alpha) \to \alpha$ in simple type theory ($\lambda{\to}$) à la Church and give a typing derivation of

$$M : ((((\alpha \to \alpha) \to \alpha) \to \alpha) \to \alpha) \to \alpha.$$

(20) 3. Consider the following two terms

- $\lambda x . x\,(\lambda y . x\,(\lambda z . y))$
- $\lambda x . x\,(\lambda y . y\,(\lambda z . x))$

For each of these terms, compute its principle type in simple type theory ($\lambda{\to}$) à la Curry, if it exists.

Give the end result and show your computation; if the term has no principle type, show how your computation yields ‘fail’.

(10) 4. We are in $\lambda 2$ à la Curry. Give a type to the term

$$\lambda x . x\,(\lambda y . y\,y)\,(\lambda z . z\,z)$$

Also give the typing derivation of your result.

Continue on the other side

5. In $\lambda 2$ à la Church we define the type B of *binary trees with node labels in* N as follows.

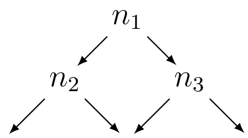
$$B \;:=\; \forall\alpha{:}*.\alpha \to (N \to \alpha \to \alpha \to \alpha) \to \alpha$$

The constant $\mathsf{leaf} : B$, creating a tree consisting of just a leaf, and the function $\mathsf{join} : N \to B \to B \to B$, joining two labelled binary trees, are defined as follows.

$$\mathsf{leaf} \;:=\; \lambda\alpha{:}*.\lambda l{:}\alpha.\lambda j{:}N \to \alpha \to \alpha \to \alpha.l$$

$$\mathsf{join} \;:=\; \lambda n{:}N.\lambda t_1, t_2{:}B.\lambda\alpha{:}*.\lambda l{:}\alpha.\lambda j{:}N \to \alpha \to \alpha \to \alpha.j\, n\, (t_1\, \alpha\, l\, j)\, (t_2\, \alpha\, l\, j).$$

(10) (a) Given $n_1, n_2, n_3 : N$, define a term t that represents the labelled binary tree in the following diagram



(10) (b) Let $n : N$ be given. Define a term $f : B \to B$ that takes a tree t and replaces all node-labels in t by n.

(15) 6. In the system λP, give a term r of type $\Pi x, y{:}A.R\, x\, y \to R\, x\, x$ in the context

$$\begin{aligned}\Gamma \;:=\;& A : *, R : A \to A \to *,\\ & t : \Pi x, y, z{:}A.R\, x\, y \to R\, y\, z \to R\, x\, z, s : \Pi x, y{:}A.R\, x\, y \to R\, y\, x.\end{aligned}$$

That is, solve the following type inhabitation problem in λP:

$$\Gamma \vdash ? : \Pi x, y{:}A.R\, x\, y \to R\, x\, x$$

Also give a typing derivation in short version (where you don't have to establish the well-formedness of the types themselves).

7. Given $Q_1, Q_2 : A \to *$, we define, in the Calculus of Constructions λC,

$$R := \lambda a{:}A.\Pi P{:}A \to *.(\Pi x{:}A.Q_1\, x \to Q_2\, x \to P\, x) \to P\, a$$

We claim that R is the "smallest set containing Q_1 and Q_2". In order to show this:

(7) (a) Give a term of type

$$\Pi a{:}A.Q_1\, a \to Q_2\, a \to R\, a$$

(8) (b) Give a term of type

$$\Pi a{:}A.R\, a \to Q_1\, a$$

NB. In both cases, give a typing derivation in short version (where you don't have to establish the well-formedness of the types themselves).

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