1. Show that the following term \( P \) is typable in simple type theory (\( \lambda \rightarrow \)) à la Church.

\[
\lambda x : (\alpha \rightarrow \beta) \rightarrow \beta \rightarrow \alpha . \lambda y : \beta . x (\lambda z : \alpha . y) y.
\]

Give a typing derivation that gives the type of \( P \).

2. Construct a term \( M \) of type \(((\alpha \rightarrow \alpha) \rightarrow \alpha) \rightarrow \alpha \) in simple type theory (\( \lambda \rightarrow \)) à la Church and give a typing derivation of

\[
M : (((\alpha \rightarrow \alpha) \rightarrow \alpha) \rightarrow \alpha) \rightarrow \alpha.
\]

3. Consider the following two terms
   - \( \lambda x . x (\lambda y . x (\lambda z . y)) \)
   - \( \lambda x . x (\lambda y . y (\lambda z . x)) \)

For each of these terms, compute its principle type in simple type theory (\( \lambda \rightarrow \)) à la Curry, if it exists.

Give the end result and show your computation; if the term has no principle type, show how your computation yields ‘fail’.

4. We are in \( \lambda 2 \) à la Curry. Give a type to the term

\[
\lambda x . x (\lambda y . y y) (\lambda z . z z)
\]

Also give the typing derivation of your result.
5. In λ2 à la Church we define the type $B$ of binary trees with node labels in $N$ as follows.

$$B := \forall \alpha : * . \alpha \to (N \to \alpha \to \alpha \to \alpha) \to \alpha$$

The constant leaf : $B$, creating a tree consisting of just a leaf, and the function join : $N \to B \to B \to B$, joining two labelled binary trees, are defined as follows.

$$\text{leaf} := \lambda \alpha : * . \lambda l : \alpha . \lambda j : N \to \alpha \to \alpha \to \alpha. l$$

$$\text{join} := \lambda n : N. \lambda t_1, t_2 : B. \lambda \alpha : * . \lambda l : \alpha . \lambda j : N \to \alpha \to \alpha \to \alpha. j n (t_1 \alpha l j) (t_2 \alpha l j).$$

(a) Given $n_1, n_2, n_3 : N$, define a term $t$ that represents the labelled binary tree in the following diagram

```
     n1
    /   \
  n2   n3
```

(b) Let $n : N$ be given. Define a term $f : B \to B$ that takes a tree $t$ and replaces all node-labels in $t$ by $n$.

6. In the system $\lambda P$, give a term $r$ of type $\Pi x, y : A. R x y \to R x x$ in the context

$$\Gamma := A : *, R : A \to A \to *, t : \Pi x, y, z : A. R x y \to R y z \to R x z, s : \Pi x, y : A. R x y \to R y x.$$

That is, solve the following type inhabitation problem in $\lambda P$:

$$\Gamma \vdash ? : \Pi x, y : A. R x y \to R x x$$

Also give a typing derivation in short version (where you don’t have to establish the well-formedness of the types themselves).

7. Given $Q_1, Q_2 : A \to *$, we define, in the Calculus of Constructions $\lambda C$,

$$R := \lambda a : A. \Pi P : A \to * . (\Pi x : A. Q_1 x \to Q_2 x \to P x) \to P a$$

We claim that $R$ is the “smallest set containing $Q_1$ and $Q_2$”. In order to show this:

(a) Give a term of type

$$\Pi a : A. Q_1 a \to Q_2 a \to R a$$

(b) Give a term of type

$$\Pi a : A. R a \to Q_1 a$$

NB. In both cases, give a typing derivation in short version (where you don’t have to establish the well-formedness of the types themselves).

END