Recall the following terms in untyped $\lambda$-calculus

- $c_n$ denotes the $n$-th Church numeral, so in particular $c_0 = \lambda f.x.x$, $c_1 = \lambda f.x.f x$ and in general $c_n = \lambda f.x.f^n(x)$.
- $K = \lambda x.y.x$, $I = \lambda x.x$

1. (a) Prove that the equation $c_0 = c_1$ is inconsistent in untyped $\lambda$ calculus.
   
   (b) (More challenging) Prove that $c_0 = c_{n+1}$ is inconsistent for any $n \in \mathbb{N}$ and conclude that $c_n = c_m$ is inconsistent for any $n, m \in \mathbb{N}$ with $n \neq m$.

2. If we take the applicative structure $(A, \text{App})$, with $A = \mathbb{N}$ (the natural numbers) and $\text{App} = \ast$ (multiplication), this cannot be made into a model of the untyped $\lambda$-calculus. Prove this.
   Hint: assume that there is an interpretation $[\cdot]$ satisfying the definition that we have given in the lecture (see Definition 59 of Berline). Now:
   
   (a) Show that $[K I] = [I K]$.
   
   (b) Conclude that $d = e$ for all $d, e \in \mathbb{N}$.
   
   So: contradiction.

3. We consider the model definition as explained in the lecture. (See Definition 57 of Berline; so we assume that the interpretation $[\cdot]$ is well-defined.)
   
   Assume $G \circ A = \text{id}_M$. Show that the $\eta$-rule holds in the model. ($\lambda x.N x = N$ for all $N$ with $x \notin \text{FV}(N)$.)

4. Prove that the theory that equates all $\lambda$-terms that don’t have a normal form is inconsistent by showing that the following equation is inconsistent in untyped $\lambda$ calculus:

$$\lambda x.y.x\,y\,\Omega = \lambda x.y.y\,x\,\Omega.$$