Semantics and Domain theory

Exercises 1

At the lecture, we gave a denotational semantics for the language \( L \) given by the grammar

\[
\begin{align*}
  b & : \text{bit} ::=: 0 \mid 1 \\
  n & : \text{bin} ::=: b \mid n \ b 
\end{align*}
\]

NB 0 and 1 are symbols, not the numbers.

The semantics is given by the model \( \mathbb{N} \), the natural numbers, and the interpretation

\[
\begin{align*}
  [0] & := 0 \\
  [1] & := 1 \\
  [n \ b] & := 2 \ast [n] + [b]
\end{align*}
\]

In the lecture, we have recursively defined the operation \( O(n) \), which prefixes a binary numeral \( n \) with a leading 0 as follows.

\[
\begin{align*}
  O(0) & := \mathbf{00} \\
  O(1) & := \mathbf{01} \\
  O(n \ b) & := O(n) \ b
\end{align*}
\]

We have given an operational semantics \( \Rightarrow \) via the rules

\[
\begin{align*}
  0 \Rightarrow \mathbf{00} & \quad 1 \Rightarrow \mathbf{01} & \quad n \Rightarrow m \\
  n \ b \Rightarrow m \ b
\end{align*}
\]

Exercises:

1. Define the operation \( S(n) \), which computes the binary numeral which is the successor of \( n \).

2. (a) Give an operational semantics for \( S(n) \), in the form of a relation \( n \Rightarrow m \) such that \( S(n) = \mathbf{m} \) iff \( n \Rightarrow m \)
   
   (b) Prove that \( S(n) = \mathbf{m} \) iff \( n \Rightarrow m \)

3. Prove \([S(n)] = [n] + 1\) for all \( n \).

4. (a) Compute the denotational semantics of \( S_1 := x := x + 1; \ y := x + x \)
   
   (b) Compute the denotational semantics of \( S_2 := \text{if } x > 0 \text{ then } x := 1 \text{ else } x := -1 \)

NB Your answer should be a "state transformers", i.e. an element of \( \text{State} \rightarrow \text{State} \), the set of partial functions from \( \text{State} \) to \( \text{State} \). For us a state is a function from variables to integers, \( r : \mathcal{V} \rightarrow \mathbb{Z} \).