Semantics and Domain theory

Exercises 10

1. (Exercise 8.4.1) Suppose that a monotonic function $p : (\mathbb{B}_\bot \times \mathbb{B}_\bot) \to \mathbb{B}_\bot$ satisfies

   - $p(\mathsf{tt}, \bot) = \mathsf{tt}$,
   - $p(\bot, \mathsf{tt}) = \mathsf{tt}$,
   - $p(\mathsf{ff}, \mathsf{ff}) = \mathsf{ff}$.

   Show that p coincides with the parallel-or function on Slide 45 in the sense that $p(d_1, d_2) = \mathrm{por}(d_1)(d_2)$, for all $d_1, d_2 \in \mathbb{B}_\bot$.

2. Prove that $\mathbf{fn}\, x : \mathbf{nat}.\, \mathbf{succ}(\mathbf{pred}\, x) \leq_{\mathrm{ctx}} \mathbf{fn}\, x : \mathbf{nat}.x$ in the following two ways:

   (a) By using the Proposition on Slide 43 directly.

   (b) By first using the Extensionality properties on Slide 44.

3. (Exercise 7.4.1.) For any PCF type τ and closed terms M_1, M_2 of type τ, show that

   $$(\forall V : \tau, (M_1 \Downarrow_\tau V \Leftrightarrow M_2 \Downarrow_\tau V)) \Rightarrow M_1 \cong_{\mathrm{ctx}} M_2 : \tau. \qquad (**)$$

   [Hint: combine the Proposition on Slide 43 with Exercise 2b of week 9 (or Lemma 7.2.1(iii)).]

4. (Exercise 7.4.2.) Use the previous exercise to show that β-conversion is valid up to contextual equivalence in PCF, in the sense that for all $\mathbf{fn}\, x : \tau_1.\, M_1 : \tau_1 \to \tau_2$ and $M_2 : \tau_1$,

   $$(\mathbf{fn}\, x : \tau_1.\, M_1)\, M_2 \cong_{\mathrm{ctx}} M_1[M_2/x] : \tau_2.$$

5. (Exercise 7.4.3.) Is the converse of (**) valid at all types?
   [Hint: recall the extensionality property of $\leq_{\mathrm{ctx}}$ at function types (Slide 44) and consider the terms $\mathbf{fix}(\mathbf{fn}\, f : \mathbf{nat} \to \mathbf{nat}.f)$ and $\mathbf{fn}\, x : \mathbf{nat}.\mathbf{fix}(\mathbf{fn}\, x : \mathbf{nat}.x)$ of type $\mathbf{nat} \to \mathbf{nat}$.]