Semantics and Domain theory

Exercises 11

Recall the following terms in untyped λ-calculus

- c_n denotes the n-th Church numeral, so in particular $c_0 = \lambda f\, x.x$, $c_1 = \lambda f\, x.f\, x$ and in general $c_n = \lambda f\, x.f^n(x)$.
- $\mathbf{K} = \lambda x\, y.x$, $\mathbf{I} = \lambda x.x$

1. (a) Prove that the equation $c_0 = c_1$ is inconsistent in untyped λ calculus.

   (b) (More challenging) Prove that $c_0 = c_{n+1}$ is inconsistent for any $n \in \mathbb{N}$ and conclude that $c_n = c_m$ is inconsistent for any $n, m \in \mathbb{N}$ with $n \neq m$.

2. The applicative structure (A, App), with $A = \mathbb{N}$ (the natural numbers) and $\mathsf{App} = *$ (multiplication) cannot be made into a (consistent) model of the untyped λ-calculus. We prove this in the following steps:

   Consider $(\mathbb{N}, *)$ and assume that there is an interpretation $[\![-]\!]$ satisfying the definition that we have given in the lecture (see Definition 59 of Berline). Now:

   (a) Show that $[\![\mathbf{K}\,\mathbf{I}]\!] = [\![\mathbf{I}\,\mathbf{K}]\!]$.

   (b) Conclude that $d = e$ for all $d, e \in \mathbb{N}$.

   So: contradiction.

3. We consider the model definition as explained in the lecture. (See Definition 57 of Berline; so we assume that the interpretation $[\![-]\!]$ is well-defined.)
   Assume $G \circ A = \mathrm{id}_M$. Show that the η-rule holds in the model. ($\lambda x.N\, x = N$ for all N with $x \notin \mathrm{FV}(N)$.)

4. Prove that the theory that equates all λ-terms that don't have a normal form is inconsistent by showing that the following equation is inconsistent in untyped λ calculus:

$$\lambda\, x\, y.x\, y\, \Omega = \lambda\, x\, y.y\, x\, \Omega.$$