Semantics and Domain theory

Exercises 4

1. Let $(D, \sqsubseteq)$ be the domain of finite and infinite sequences over $\Sigma := \{a, b\}$ with $\sqsubseteq$ the prefix ordering. (So $D = \Sigma^* \cup \Sigma^\omega$.)
   (a) Which of the following functions $f : D \to D$ is monotone / continuous?
      i. $f(s) = s$ with all a's removed.
      ii. $f(s) = abba$ if s is finite; $f(s) = s$ if s is infinite.
      iii. $f(s) = abbas$.
      iv. $f(s) = a$ if s contains finitely many b's; $f(s) = b$ if s contains infinitely many b's
   (b) For each of the functions f in (a) that is continuous, compute the least fixed point of f.

2. Let $(D, \sqsubseteq)$ be a domain with some element d_0 and let $f : D \to D$ be continuous. Suppose $d_0 \sqsubseteq f(d_0)$. Prove that $\sqcup_{i \geq 0} f^i(d_0)$ is a fixed point of f.

3. For the *disjoint union* of two domains (also called the *binary sum* of domains), there are two choices: the *coalesced sum* (or *smashed sum*) $D +_c E$, or the *separated sum* $D +_s E$.

   For the coalesced sum, the set $D +_c E$ is defined as

$$\{\bot\} \cup \{(0, d) \mid d \in D, d \neq \bot_D\} \cup \{(1, e) \mid e \in E, e \neq \bot_E\}$$

   For the separated sum, the set $D +_s E$ is defined as

$$\{\bot\} \cup \{(0, d) \mid d \in D\} \cup \{(1, e) \mid e \in E\}$$

   So, the separated sum introduces a new $\bot$ element, whereas the coalesced sum "coalesces (or smashes) them together".

   Let two domains $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$ be given.

   (a) Define the partial ordering $\sqsubseteq$ on $D +_s E$ and give the $\bot$-element.
   (b) Define the partial ordering $\sqsubseteq$ on $D +_c E$ and give the $\bot$-element.
   (c) For $(f_i)_{i \geq 0}$ a chain in $D +_s E$ define $\sqcup_{i \geq 0} f_i$ and prove that it is the least upperbound.
   (d) For $(f_i)_{i \geq 0}$ a chain in $D +_c E$ define $\sqcup_{i \geq 0} f_i$ and prove that it is the least upperbound.
   (e) Define injections $\mathsf{inl} : D \to D +_s E$ and $\mathsf{inr} : E \to D +_s E$ that are continuous. (You don't have to prove that they are continuous.)
   (f) Define injections $\mathsf{inl} : D \to D +_c E$ and $\mathsf{inr} : E \to D +_c E$ that are continuous. (You don't have to prove that they are continuous.)
   (g) (*) For F a domain and $f : D \to F$, $g : E \to F$ we want to define a continuous function $[f, g] : D + E \to F$ such that $[f, g](\mathsf{inl}(x)) = f(x)$ and $[f, g](\mathsf{inr}(x)) = g(x)$.
   Show how to define $[f, g]$ for the case of $D +_c E$ and for the case of $D +_s E$. For one of these cases, we can only define $[f, g]$ if we place additional requirements on f and g. Which?