Semantics and Domain theory
Exercises 4

1. Let \((D, \sqsubseteq)\) be the domain of finite and infinite sequences over \(\Sigma := \{a, b\}\) with \(\sqsubseteq\) the prefix ordering. (So \(D = \Sigma^* \cup \Sigma^\omega\).)
   
   (a) Which of the following functions \(f : D \rightarrow D\) is monotone / continuous?
   
   i. \(f(s) = s\) with all \(a\)'s removed.
   
   ii. \(f(s) = abba\) if \(s\) is finite; \(f(s) = s\) if \(s\) is infinite.
   
   iii. \(f(s) = abbas\).
   
   iv. \(f(s) = a\) if \(s\) contains finitely many \(b\)'s; \(f(s) = b\) if \(s\) contains infinitely many \(b\)'s

   (b) For each of the functions \(f\) in (a) that is continuous, compute the least fixed point of \(f\).

2. Let \((D, \sqsubseteq)\) be a domain with some element \(d_0\) and let \(f : D \rightarrow D\) be continuous. Suppose \(d_0 \sqsubseteq f(d_0)\). Prove that \(\sqcup_{i \geq 0} f^i(d_0)\) is a fixed point of \(f\).

3. For the disjoint union of two domains (also called the binary sum of domains), there are two choices: the coalesced sum (or smashed sum) \(D +_c E\), or the separated sum \(D +_s E\).

   For the coalesced sum, the set \(D +_c E\) is defined as
   
   \[
   \{\bot\} \cup \{(0, d) \mid d \in D, d \neq \bot_D\} \cup \{(1, e) \mid e \in E, e \neq \bot_E\}
   \]

   For the separated sum, the set \(D +_s E\) is defined as
   
   \[
   \{\bot\} \cup \{(0, d) \mid d \in D\} \cup \{(1, e) \mid e \in E\}
   \]

   So, the separated sum introduces a new \(\bot\) element, whereas the coalesced sum “coalesces (or smashes) them together”.

   Let two domains \((D, \sqsubseteq_D)\) and \((E, \sqsubseteq_E)\) be given.

   (a) Define the partial ordering \(\sqsubseteq\) on \(D +_s E\) and give the \(\bot\)-element.

   (b) Define the partial ordering \(\sqsubseteq\) on \(D +_c E\) and give the \(\bot\)-element.

   (c) For \(\{f_i\}_{i \geq 0}\) a chain in \(D +_s E\) define \(\sqcup_{i \geq 0} f^i\) and prove that it is the least upperbound.

   (d) For \(\{f_i\}_{i \geq 0}\) a chain in \(D +_c E\) define \(\sqcup_{i \geq 0} f^i\) and prove that it is the least upperbound.

   (e) Define injections \(\text{inl} : D \rightarrow D +_s E\) and \(\text{inr} : E \rightarrow D +_s E\) that are continuous. (You don’t have to prove that they are continuous.)

   (f) Define injections \(\text{inl} : D \rightarrow D +_c E\) and \(\text{inr} : E \rightarrow D +_c E\) that are continuous. (You don’t have to prove that they are continuous.)

   (g) (*) For \(F\) a domain and \(f : D \rightarrow F, g : E \rightarrow F\) we want to define a continuous function \([f, g] : D + E \rightarrow F\) such that \([f, g](\text{inl}(x)) = f(x)\) and \([f, g](\text{inr}(x)) = g(x)\).

   Show how to define \([f, g]\) for the case of \(D +_c E\) and for the case of \(D +_s E\). For one of these cases, we can only define \([f, g]\) if we place additional requirements on \(f\) and \(g\). Which?