Semantics and Domain theory
Exercises 6

1. (This is basically Exercise 4.4.1. of DENS)
   
   (a) Show that if $S$ and $T$ are chain-closed, then $S \cup T$ is chain-closed.
   
   (b) Show that if $S_i$ is chain-closed for every $i \in I$, then $\bigcap_{i \in I} S_i$ is chain-closed.

2. (Exercise 4.4.2.) Give an example of a subset $S$ of $D \times D$ that is not chain-closed, but which satisfies:
   
   (a) $\forall d \in D, \{d' | (d, d') \in S\}$ is chain-closed
   
   (b) $\forall d' \in D, \{d | (d, d') \in S\}$ is chain-closed.

   [Hint: consider $D = D = \Omega$, the cpo in Figure 1.] (Compare this with the property of continuous functions given on Slide 15.)

3. The collection of chain-closed sets is not closed under arbitrary union. (It is not the case, in general that $\forall i \in I (S_i \text{ is chain closed})$ implies $\bigcup_{i \in I} S_i$ is chain-closed.)

   (a) Conclude this from the previous exercise.
   
   (b) Conclude this by directly constructing a counterexample in $\Omega$.

4. Let $P : D \to B_\bot$ and $g : D \to D$ be continuous. Define $f : D \times D \to D \times D$ by
   
   $$f(d_1, d_2) = \text{if}(P(d_1), (g(d_1), g(d_2)), (g(d_2), g(d_1))).$$

   Show that for fix($f$) = $(u_1, u_2)$, we have $u_1 = u_2$. (Use Scott induction.)

5. Prove that for $f : D \to E$ monotone,

   $$f^{-1} \text{ preserves chain-closed sets} \Rightarrow f \text{ is continuous},$$

   where $f^{-1} \text{ preserves chain-closed sets}$ means that, for all $S \subseteq E$, if $S$ is chain-closed, then $f^{-1}(S)$ is a chain-closed subset of $D$. 