Semantics and Domain theory
Exercises 8

1. (a) Give a type \( \tau \), a term \( M \), values \( V, V' \) and a context \( C[-] \) such that \( M \Downarrow\tau V \) but \( C[M] \Downarrow\tau V' \neq C[V] \).
   (b) Give a type \( \tau \), a term \( M \), a value \( V \) and a context \( C[-] \) such that \( M \Downarrow\tau V \) but \( C[M] \nmid\tau \)
   \( (C[M] \text{ has no value}) \)
   (c) Give a type \( \tau \), a term \( M \), a value \( V \) and a context \( C[-] \) such that \( M \nmid\tau \) but \( C[M] \Downarrow\tau V \)

2. Prove that the following terms \( M \) and \( N \) are not contextually equivalent.
   (a) \( M = \text{if } x \text{ then } 0 \text{ else } 1 \) and \( N = \text{if } y \text{ then } 0 \text{ else } 1 \).
   (b) \( M = \text{fn } x : \text{nat. succ(pred } x) \) and \( N = \text{fn } x : \text{nat } . x \).

3. (Exercise 6.5.2.) Define \( \Omega_{\tau} = \text{fix}(\text{fn } x : \tau.x) \)
   (a) Show that \( \downarrow[\Omega_{\tau}] \) is the least element of the domain \( [\tau] \).
   (b) Deduce that \( \downarrow[\text{fn } x : \tau.\Omega_{\tau}] = \downarrow[\Omega_{\tau\rightarrow\tau}] \).

4. (a) Compute the denotational semantics of \( M = \text{fn } x : \text{bool.fn } y : \text{nat.if } x \text{ then } y \text{ else } y \)
   (b) Define a term \( N \) such that \( \downarrow[M] = \downarrow[N] \) but \( N \nmid M \).

5. Define terms \( M, N : \text{nat } \rightarrow \text{nat} \) with \( \downarrow[M] \sqsubseteq \downarrow[N] \) and \( \downarrow[M] \neq \downarrow[N] \).

6. Verify that \( \downarrow[(\text{fn } x : \sigma. M)N] = \downarrow[M[N/x]] \) for \( M, N \) with \( \vdash N : \sigma \) and \( x : \sigma \vdash M : \tau \). (Use the result on Slide 38, the Substitution Lemma.)