# Semantics and Domain theory

**Exercises 12**

1. Which of the following sets are complete lattices.
   (a) The set of flat natural numbers $\mathbb{N}_\bot$.
   (b) The set $\mathcal{P}_{\text{fin}}(\mathbb{N})$ of *finite subsets* of $\mathbb{N}$.
   (c) The set $\Omega$ (= $\mathbb{N} \cup \{\omega\}$, with the ordering we have seen before).
   (d) The set of monotone functions from $\mathbb{B}_\bot^\top$ to $\mathbb{B}_\bot^\top$.
   (Remember that the set of flat booleans with a top element added, $\mathbb{B}_\bot^\top$, is a complete lattice.)

2. Complete the proof of Proposition 3.1.7.
   That is, show that in a complete lattice $(D, \sqsubseteq)$, if we define

$$\bigsqcap X := \bigsqcup \{y \in D \mid y \sqsubseteq X\},$$

   then $\bigsqcap X$ is indeed the *greatest lower bound* (also called the *inf*) of $X$.

3. Prove the correctness of Definition 3.2.5. To prove this, you have to show that the function

$$\boldsymbol{\lambda} d.[\![P]\!]_{\rho(x:=d)}$$

   is continuous for every $P$ and $\rho$. (You may assume that $F$ and $G$ are continuous and all the other results about continuity from the notes.)

4. At the lecture, we have seen the interpretations in $D_A$ of $\mathbf{I}$ (= $\lambda x.x$), $\mathbf{K}$ (= $\lambda x.\lambda y.x$) and $\mathbf{II}$.
   (a) Compute the interpretation of $\lambda x.x\,x$.
   (b) Show that $[\![\mathbf{KI}]\!] = \{(\beta, (\gamma, c)) \mid c \in \gamma\}$ (without doing a $\beta$-reduction first).

5. Let $Y$ be an element of $D_A$ and let $\rho$ be a valuation with $\rho(y) = Y$.
   (a) Compute in $D_A$ the interpretation of $\lambda x.y\,x$ by expressing $[\![\lambda x.y\,x]\!]_\rho$ in terms of $Y$.
   (b) Conclude that the $\eta$-rule does not hold in $D_A$. (The $\eta$-rule says that $\lambda x.M\,x = M$ if $x \notin \text{FV}(M)$.)

6. Use the result of the following exercise ($[\![\Omega]\!] = \emptyset$) to
   (a) compute the interpretation of $\lambda y.\Omega$ in $D_A$,
   (b) compute the interpretation of $\lambda y.y\,\Omega$ in $D_A$.

7. [Challenging] Show that the interpretation of $\Omega$ (= $(\lambda x.x\,x)(\lambda x.x\,x)$ ) in $D_A$ is $\emptyset$.
   (Hint: From a $c \in [\![\Omega]\!]$ you can construct an infinite sequence $(\alpha_i)_{i\in\mathbb{N}}$ with $(\alpha_{i+1}, c) \in \alpha_i$ for all $i$, which is impossible in $D_A$.)